XVIII. *On* RICCATI'S *Equation and its Transformations, and on some Definite Integrals which satisfy them.*

By J. W. L. GLAISHER, *M.A., F.R.S., Fellow of Trinity College, Cambridge.*

Received and read June 16, 1881.

Introduction.

THE present memoir relates chiefly to the different forms of the particular integrals of the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p+1)}{x^2}u \quad \ldots \ldots \ldots \ldots (1),$$

and to the evaluation of certain definite integrals which are connected with this equation. Transforming (1) by assuming $u=x^{-p}v$, it becomes

$$\frac{d^2v}{dx^2}-\frac{2p}{x}\frac{dv}{dx}-a^2v=0 \quad \ldots \ldots \ldots \ldots (2),$$

that is, writing $n-1$ for $2p$,

$$\frac{d^2v}{dx^2}-\frac{n-1}{x}\frac{dv}{dx}-a^2v=0 \quad \ldots \ldots \ldots \ldots (3),$$

and this equation may be transformed into

$$\frac{d^2v}{dz^2}-a^2z^{2q-2}v=0 \quad \ldots \ldots \ldots \ldots (4),$$

by the substitution $x=\frac{1}{q}z^q$, where $q=\frac{1}{n}$. The equation (4) may be regarded as the standard form of RICCATI'S equation (see § III., art. 17).

It is well-known that these equations admit of integration in a finite form if $p=$ an integer, $n=$ an uneven integer, and $q=$ the reciprocal of an uneven integer, respectively.

The contents of the memoir are as follows:

In the first section (§ I.) six particular integrals of the equation (1) are obtained, and the relations between them are examined. When p is not an integer, all the six integrals extend to infinity, and in this case the relations between them present no special peculiarity. When p is an integer, two of the series terminate, and we thus obtain two particular integrals of (1) which contain a finite number of terms. The series terminate in consequence of the occurrence of zero factors in the coefficients of the terms, but if they are continued, zero factors occur also in the denominators, so that, after a finite number of zero terms, the series may be regarded as recommencing and extending to infinity. If the terminating series are supposed to recommence in this manner, so that all the series extend to infinity, then the relations between the particular integrals are the same as when p is not an integer; but if the series are supposed to terminate absolutely when the zero terms occur, the relations are quite different. As the finite portions of the particular integrals satisfy the differential equation, it is more natural to regard the series as terminating absolutely, and on this supposition the relations between the particular integrals exhibit a remarkable diversity of form according as p is or is not an integer.

The second section contains what is believed to be a new form of the solution of (1) in the case of p = an integer. It is shown that if $p=i$, a positive integer, this equation is satisfied by the coefficient of h^{i+1} in the expansion of $e^{a\sqrt{(x^2+xh)}}$ in ascending powers of h. The six particular integrals given in § I. of the equation (1) and the relations connecting them are obtained by different expansions of this expression.

The third section contains the six particular integrals of (3) and (4) corresponding to those of (1), from which they are deduced by means of the transformations stated above.

The fourth section relates to the particular cases in which the differential equations admit of integration in a finite form. If a differential equation is satisfied by an infinite series, and if for certain values of a quantity involved in it the series terminates, then in this case we may present the integral in a different form by commencing the finite series at the other end, and writing the terms in the reverse order.

Thus, for example, a particular integral of (1) is $u=\mathrm{P}$, where

$$\mathrm{P}=x^{-p}\left\{1-\frac{p}{p}ax+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{a^2x^2}{2\,!}-\frac{p(p-1)(p-2)}{p(p-\frac{1}{2})(p-1)}\frac{a^3x^3}{3\,!}+\&\text{c.}\right\}e^{ax},$$

but, if p = a positive integer, then commencing the series at the other end,

$$\mathrm{P}=(-)^p\frac{2^pa^p}{(p+1)(p+2)\ldots 2p}\left\{1-\frac{p(p+1)}{2}\frac{1}{ax}+\frac{(p-1)p(p+1)(p+2)}{2.4}\frac{1}{a^2x^2}\cdots\right.$$
$$\left.+(-)^p\frac{1.2\ldots 2p}{2.4\ldots 2p}\frac{1}{a^px^p}\right\}e^{ax}.$$

These reverse forms in the case of the equations (1), (3), (4) are given in this section.

It is worthy of remark that if we are given a particular integral of a differential equation in the form of a terminating series, such as, for example,

$$1-\frac{p(p+1)}{2}\frac{1}{ax}+\frac{(p-1)p(p+1)(p+2)}{2.4}\frac{1}{a^2x^2}-\&c.,$$

p being a positive integer, then we might suppose that the corresponding particular integral, when p was not an integer, would be obtained by continuing the series, which does not then terminate, to infinity. This infinite series, when p is not an integer, still satisfies the differential equation, but is divergent; and the true integral is obtained by commencing the series at the other end and continuing it to infinity backwards. In general, when we have a series which terminates of itself for a particular form of p, we may derive from it two infinite series, when p has not this form, by commencing it at either end. One of these will be an ascending series and the other a descending series; and we can thus, as it were, pass from the one to the other through the intervention of the finite series.

The fifth section contains the evaluations of the definite integrals

$$\int_0^\infty e^{-x^m-\frac{a^2}{x^m}}dx, \qquad \int_0^\infty \frac{\cos bx}{(a^2+x^2)^n}dx,$$

where m denotes any real quantity and n any positive quantity. These integrals have been evaluated when m is of the form $\frac{-4i}{2i\pm1}$ and when n is a positive integer; but, so far as I know, the general formulæ given are new. It is known that these integrals satisfy differential equations of the forms (4) and (1) respectively, so that their values are necessarily connected with the solutions of these equations considered in §§ I. and III. The results are curious, as they exhibit changes of form similar to those referred to in describing the contents of § I., and which are due to the same cause—viz. the recommencement of the terminating series after the zero terms.

When n is unrestricted it is shown that we have

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{a^2}{x^2}}dx= \tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\left\{1+\frac{n-1}{n-1}(2a)+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{(2a)^2}{2\,!}+\&c.\right\}e^{-2a}$$
$$+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)a^n\left\{1+\frac{n+1}{n+1}(2a)+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{(2a)^2}{2\,!}+\&c.\right\}e^{-2a};$$

but when n is a positive integer the first series is to be continued till it terminates, and the second is to be ignored; and if n is a negative integer the second is to be continued till it terminates, and the first is to be ignored. The well-known value of the integral when $n-1$ = an even integer $=2i$, viz.

$$\int_0^\infty x^{2i}e^{-x^2-\frac{a^2}{x^2}}dx=\frac{\sqrt{\pi}}{2}a^i\left\{1+\frac{i(i+1)}{2}\frac{1}{2a}+\frac{(i-1)i(i+1)(i+2)}{2.4}\left(\frac{1}{2a}\right)^2+\&c.\right\}e^{-2a},$$

does not suggest the general formula, the terms of the finite series being written in the reverse order.

Certain formulæ of BOOLE'S and CAUCHY'S are also considered and extended in this section.

The sixth section, which is the longest in the memoir, relates to the numerous symbolic solutions of the equation (1) and its transformations (3) and (4) in the cases in which they are integrable in finite terms. In this section these symbolic solutions are derived from the definite integrals considered in § V.; and the various symbolic theorems to which they lead by comparing different forms of the results are examined. A great many symbolic solutions of the differential equations have been given by R. L. ELLIS, BOOLE, LEBESGUE, HARGREAVE, WILLIAMSON, DONKIN, &c., and these are briefly noticed and connected with one another. It may be observed that the solution

$$u=x^{p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{p+1}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right),$$

which has been several times independently discovered, seems to have been first published by Mr. GASKIN, who in effect gave it in a problem set in the Senate House Examination at Cambridge in 1839.

The seventh section relates to the connexion between the results given in §§ I.–VI. and the formulæ of BESSEL'S Functions. BESSEL'S equation

$$\frac{d^2w}{dx^2}+\frac{1}{x}\frac{dw}{dx}+\left(1-\frac{\nu^2}{x^2}\right)w=0,$$

may be derived from (1) by the simple substitutions

$$u=x^{\frac{1}{2}}w,\quad p+\tfrac{1}{2}=\nu,\quad a^2=-1\,;$$

so that all the theorems relating to the solutions of (1) have analogues in the solutions of BESSEL'S equation, which are deducible from them by these transformations. In this section the formulæ in BESSEL'S Functions which correspond to those considered in the memoir are stated in a convenient form for comparison. The number of such formulæ is not great, and the substitution of $\sqrt{(-1)}$ for a, which converts exponentials into sines and cosines, and a single series multiplied by an exponential factor into the sum of two series multiplied respectively by a sine and a cosine, changes considerably the appearance of the results, which, from an analytical point of view, are less simple when the differential equation is of BESSEL'S form. The principal case considered in the theory of BESSEL'S Functions is that of ν = an integer: this corresponds to the case of p = an integer $+\frac{1}{2}$, which is generally excluded in this memoir, as it renders certain of the particular integrals infinite (§ I., arts. 1, 3). The case of finite solution corresponds in (1) to p = an integer and in BESSEL'S equation to ν = an integer $+\frac{1}{2}$. The fact that BESSEL'S function $J^{\nu}(x)$ is expressible in a finite form when $\nu=i+\frac{1}{2}$, and the finite expression itself, are well known, and the case is an important one in physical investigations; but, so far as I know, the recommencement of the series after the zero terms has not been specially noticed in connexion with the subject of BESSEL'S Functions.

The eighth (and last) section contains a list of writings the contents of which are closely connected with the subject of the memoir, arranged in order of date and classed under the sections in which they are noticed. There is also in each case a short account of the portion of the paper for which it has been referred to, with the numbers of the articles in which the references occur. The section does not contain a list of all the papers referred to in the memoir; only those papers which are closely connected with it, and portions of which are, in most cases, to some extent reproduced in it, being included. The part of the list which relates to § VI. is intended to be supplementary to that section: it is not in any sense a bibliography of the symbolic solutions, but it probably contains references to all the more important papers on the subject.

In the 'Philosophical Magazine' for 1868 CAYLEY gave the four particular integrals P_2, Q_2, R_2, S_2 (§ III.) of RICCATI'S equation (4); and in the same journal for 1872 I investigated the relations between these four particular integrals and the well-known particular integrals U_2, V_2. The results are the same as those given in § III., and the method is similar to that employed in § I. I afterwards found that the process of obtaining and connecting the particular integrals assumed a much more simple form when the differential equation was taken to be (1) than when it was (4); and it seemed desirable to re-write the whole investigation, taking (1) as the differential equation. This investigation forms § I.; it is similar in every respect to that contained in the 'Philosophical Magazine,' but is much more complete. The corresponding results for the equations (3) and (4) are deduced in § III.

The fact that, in the solution in series of a differential equation, if the series terminates but when continued recommences, the latter portion as well as the finite series satisfies the differential equation, was pointed out by CAYLEY in the 'Messenger of Mathematics' for 1869.

The formula (8) of § V. was published in the 'British Association Report' for 1872, with a brief account of the process given in arts. 20, 21. The principal portion of two short papers, "On RICCATI'S Equation" and "On certain Differential Equations allied to RICCATI'S," which were published in the 'Quarterly Journal of Mathematics' for 1871 and 1872, are incorporated in § VI.

The memoir thus includes the results contained in several scattered notes and papers. In these the differential equation considered was generally RICCATI'S in the form (4), but the advantage of adopting (1) as the standard form in preference to (4) is considerable. As far as the differential equation is concerned (4), which consists of only two terms, is the simplest form; but as regards the expression of the results, both (1) and (3) are superior in every respect. The equation (3) was adopted as the standard form by M. BACH in his paper of 1874 (see § VIII.).

The form $2q-2$ for the exponent in RICCATI's equation (4) was first employed, I believe, in CAYLEY's paper in the 'Philosophical Magazine' for 1868, which has been already referred to. The use of the quantity q greatly simplifies the formulæ relating to the solution of the equation.

With the exception of § VII., the memoir was written about three years ago, the delay in communicating it to the Society being due to the fact that it seemed desirable to connect the results more closely with BESSEL's Functions. As the theory of these functions forms a distinct and recognised branch of analysis, and as the differential equations considered are transformable into BESSEL's equation by very simple changes in the variables, it was clearly of importance to examine with some care the connexion of the formulæ with those of BESSEL's Functions, and it even seemed possible that it might be advisable to adopt BESSEL's equation as the standard form. For the reasons already stated it appeared that this was not the case, and that the analytical treatment of the subject was complicated by the change to BESSEL's equation. It is well known that the general integrals of the differential equations (1), . . . (4) can be expressed in terms of BESSEL's Functions; and LOMMEL has specially considered these solutions in several papers in the 'Mathematische Annalen.'* In these papers, however, the points to which the memoir relates are not referred to. It therefore seemed sufficient to give in § VII. the connexion between the principal formulæ, reserving for a separate paper, if it should appear desirable, the examination of the relations in which the series considered in the memoir stand to BESSEL's Functions with negative indexes and to the functions of the second kind introduced by LOMMEL and by NEUMANN.

During the time that the memoir has been in manuscript I have published two extracts from it, viz. the theorem in § II., arts. 8, 9, in the 'British Association Report' for 1880, and the theorem (50) and its proof (§ VI., art. 41) in the 'Proceedings of the Cambridge Philosophical Society' for 1879.

The differential equations (1), . . . (4) present three distinct peculiarities, viz. (i.) they are finitely integrable only in special cases; (ii.) they are satisfied by certain remarkable definite integrals, which have attracted attention quite independently of the differential equations; and (iii.) the solutions when finite admit of being exhibited in various symbolic forms. In reference to the third of these properties, it is remarkable how much attention has been devoted to the solutions of the equations in these finite cases during the last fifty years. The differential equations (1), . . . (4) have however been frequently discussed not as simple transformations of one another

* Vol. ii. (1870), pp. 624–635; vol. iii. (1871), pp. 475–487; vol. xiv. (1879), pp. 510–536.

but as if they were essentially different, and the processes of solution have been applied to them independently. Also many of the forms have been re-discovered several times; and it therefore seemed to be worth while to collect together, as in § VI., the different symbolic formulæ, and exhibit the nature of the relations between them.

Although the equation (1) is connected with BESSEL'S equation by such simple relations, the methods of treatment of the two equations by mathematicians have been very different. In the case of (1) and its transformations (2), (3), (4), the purely analytical part of the theory and the forms of the solutions have chiefly attracted attention; while in the case of BESSEL'S equation the expansion of the results in series suitable for calculation has been one of the main objects. The theories of the two equations have been developed from very different points of view: the one has been considered in reference to the methods of solution and the peculiarities already referred to, and the other has been considered almost wholly in connexion with the functions which satisfy it, and their applications in astronomy and physics. It is curious that two such very distinct classes of analytical investigation should have been formed having reference to differential equations so closely connected.

It is proper to remark here that in the differential equation (1) and throughout the memoir the constant a may be put equal to unity without loss of generality. It was found to be desirable to retain it, as there is some advantage in having present in the solutions a letter whose sign can be changed at pleasure, and also because the transition to the differential equations

$$\frac{d^2u}{dx^2}+a^2u=\frac{p(p+1)}{x^2}u, \quad \&c.,$$

(*i.e.*, in which the sign of a^2 is changed) is thus rendered somewhat more convenient.

The ordinary differential equations (1), . . . (4) are considered throughout, and no reference is made to the corresponding partial differential equations

$$\frac{d^2u}{dx^2}-a^2\frac{d^2u}{dy^2}=\frac{p(p+1)}{x^2}u, \quad \&c.,$$

the solutions of which may be deduced in the usual manner by replacing a by $a\frac{d}{dy}$, and c_1e^{ax} and c_2e^{ax} by $\phi(y+ax)$ and $\psi(y+ax)$. No point of interest arises in connexion with this transition.

Following the notation usually adopted in connexion with the differential equation (1), i is used throughout to denote a positive integer. The expression $\sqrt{(-1)}$, which occurs only towards the end of § VI. and in § VII., is denoted by i'.

The headings of the eight sections, with the numbers of the articles which they contain and the pages, are as follows:

---

§ I.

Direct integration of the differential equation in series, and connexion between the particular integrals. Arts. 1–7.

1. The most direct method of integrating the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p+1)}{x^2}u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and obtaining the relations that exist between the different particular integrals, appears to be as follows.

Let

$$u=\Sigma A_r x^{m+r},$$

the summation extending to all positive integral values of r; then, substituting in the differential equation, we have

$$(m+r+p)(m+r-p-1)A_r-a^2A_{r-2}=0,$$

whence, putting $r=0$ or 1,

$$m=-p \text{ or } p+1.$$

Taking the first root, the equations giving A_2, A_4, A_6 ... are

$$2(1-2p)A_2-a^2A_0=0,$$
$$4(3-2p)A_4-a^2A_2=0,$$
$$6(5-2p)A_6-a^2A_4=0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

whence

$$A_2 = -\frac{1}{p-\frac{1}{2}}\frac{a^2}{2^2}A_0,$$

$$A_4 = -\frac{1}{2}\frac{1}{p-\frac{3}{2}}\frac{a^2}{2^2}A_2,$$

$$A_6 = -\frac{1}{3}\frac{1}{p-\frac{5}{2}}\frac{a^2}{2^2}A_4,$$

$$\cdots\cdots\cdots$$

so that the solution corresponding to the root $m=-p$ is

$$U = x^{-p}\left\{1-\frac{1}{p-\frac{1}{2}}\frac{a^2x^2}{2^2}+\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\frac{a^4x^4}{2^4.2!}-\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})(p-\frac{5}{2})}\frac{a^6x^6}{2^6.3!}+\&c.\right\},$$

where, as throughout this memoir, $r!$ denotes $1.2.3\ldots r$.

Similarly, taking the root $m=-p-1$, the other solution is found to be

$$V = x^{p+1}\left\{1+\frac{1}{p+\frac{3}{2}}\frac{a^2x^2}{2^2}+\frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})}\frac{a^4x^4}{2^4.2!}+\frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})(p+\frac{7}{2})}\frac{a^6x^6}{2^6.3!}+\&c.\right\},$$

and, as U and V are independent, the complete integral of the differential equation is $AU+BV$, A and B being arbitrary constants.

There is nothing in the form of these series to indicate that for any values of p the integral of the differential equation admits of being expressed in a finite form. They show, however, that if p = the half of an uneven integer (the case $p=-\frac{1}{2}$ alone excepted) the solution assumes a different form, viz. if, say, in U the terms after a certain point become infinite, the solution is of the form $W+V\log cx$, W being a new series. This case is excluded in what follows; and throughout the memoir p is supposed not to be of the forms $\pm\frac{1}{2}(2n+1)$. If, however, p is of either of these forms only certain of the series considered will involve infinite terms, and the relations connecting those series which do not involve infinite terms will still remain true.

2. Transforming the differential equation (1) by assuming $u=e^{ax}v$, a substitution suggested by the form of the first member of the equation, we obtain the differential equation in v

$$\frac{d^2v}{dx^2}+2a\frac{dv}{dx}=\frac{p(p+1)}{x^2}v.$$

Putting as before

$$v=\Sigma A_r x^{m+r},$$

we have

$$(m+r+p)(m+r-p-1)A_r+2(m+r-1)aA_{r-1}=0,$$

whence

$$m=-p \text{ or } p+1.$$

Taking the first root, the equations are

$$\begin{gathered}(\ \ -2p)A_1+2(\ \ -p)aA_0=0,\\ 2(1-2p)A_2+2(1-p)aA_1=0,\\ 3(2-2p)A_3+2(2-p)aA_2=0,\\ \cdots\cdots\cdots\cdots\end{gathered}$$

giving

$$A_1=-\frac{p}{p}aA_0,\qquad A_2=-\tfrac{1}{2}\frac{p-1}{p-\frac{1}{2}}aA_1,\qquad A_3=-\tfrac{1}{3}\frac{p-2}{p-1}aA_2,\ \ldots\ ;$$

and we obtain the particular integral

$$x^{-p}\left\{1-\frac{p}{p}ax+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{a^2x^2}{2!}-\frac{p(p-1)(p-2)}{p(p-\frac{1}{2})(p-1)}\frac{a^3x^3}{3!}+\&c.\right\}.$$

Similarly, the other particular integral is found to be

$$x^{p+1}\left\{1+\frac{p+1}{p+1}ax+\frac{(p+1)(p+2)}{(p+1)(p+\frac{3}{2})}\frac{a^2x^2}{2!}+\frac{(p+1)(p+2)(p+3)}{(p+1)(p+\frac{3}{2})(p+2)}\frac{a^3x^3}{3!}+\&c.\right\}.$$

If we had transformed (1) by assuming $u=e^{-ax}v$, we should have obtained a differential equation in v differing from that given above only in having the sign of a changed: and the two particular integrals would differ from those just written only in having the signs of the alternate terms negative.

3. Thus, of the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p+1)}{x^2}u,$$

we have obtained the six particular integrals U, V, P, Q, R, S, where

$$U=x^{-p}\left\{1-\frac{1}{p-\frac{1}{2}}\frac{a^2x^2}{2^2}+\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\frac{a^4x^4}{2^4.2!}-\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})(p-\frac{5}{2})}\frac{a^6x^6}{2^6.3!}+\&c.\right\},$$

$$V=x^{p+1}\left\{1+\frac{1}{p+\frac{3}{2}}\frac{a^2x^2}{2^2}+\frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})}\frac{a^4x^4}{2^4.2!}+\frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})(p+\frac{7}{2})}\frac{a^6x^6}{2^6.3!}+\&c.\right\},$$

$$P=x^{-p}\left\{1-\frac{p}{p}ax+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{a^2x^2}{2!}-\frac{p(p-1)(p-2)}{p(p-\frac{1}{2})(p-1)}\frac{a^3x^3}{3!}+\&c.\right\}e^{ax},$$

$$Q=x^{p+1}\left\{1-\frac{p+1}{p+1}ax+\frac{(p+1)(p+2)}{(p+1)(p+\frac{3}{2})}\frac{a^2x^2}{2!}-\frac{(p+1)(p+2)(p+3)}{(p+1)(p+\frac{3}{2})(p+2)}\frac{a^3x^3}{3!}+\&c.\right\}e^{ax},$$

$$R=x^{-p}\left\{1+\frac{p}{p}ax+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{a^2x^2}{2!}+\frac{p(p-1)(p-2)}{p(p-\frac{1}{2})(p-1)}\frac{a^3x^3}{3!}+\&c.\right\}e^{-ax},$$

$$S=x^{p+1}\left\{1+\frac{p+1}{p+1}ax+\frac{(p+1)(p+2)}{(p+1)(p+\frac{3}{2})}\frac{a^2x^2}{2!}+\frac{(p+1)(p+2)(p+3)}{(p+1)(p+\frac{3}{2})(p+2)}\frac{a^3x^3}{3!}+\&c.\right\}e^{-ax}.$$

These integrals form three pairs U and V, P and Q, R and S, either of the integrals in each pair being deducible from the other by the substitution of $-(p+1)$ for p: and, since the differential equation involves p only in the form $p(p+1)$, it is evident *à priori* that if in any expression satisfying the differential equation, p is replaced by $-(p+1)$, the new expression must still satisfy the differential equation.

Also the pairs P and Q, R and S, are convertible the one into the other by changing the sign of a.

4. If p is a positive integer the series in P and R terminate and the general integral of the differential equation is AP+BR; and if p is a negative integer, the series in Q and S terminate and the general integral is AQ+BS.

Thus, if $p=2$, the general integral is

$$u=\mathrm{A}x^{-2}\{1-ax+\tfrac{1}{3}a^2x^2\}e^{ax}+\mathrm{B}x^{-2}\{1+ax+\tfrac{1}{3}a^2x^2\}e^{-ax};$$

and, if $p=-2$, the general integral is

$$u=\mathrm{A}x^{-1}\{1-ax\}e^{ax}+\mathrm{B}x^{-1}\{1+ax\}e^{-ax}.$$

5. As however we have six particular integrals, of which, for any given value of p, only two can be independent, it remains to investigate the relations between the particular integrals in the different cases that arise.

(1°.) Suppose p unrestricted (except as mentioned in art. 1), but not equal to an integer.

In this case all the series extend to infinity, and

$$\mathrm{P}=\mathrm{R}=\mathrm{U}, \qquad \mathrm{Q}=\mathrm{S}=\mathrm{V}$$

for, leaving out of consideration the factor x^{-p} that occurs in both P and U, the coefficient of a^nx^n in P

$$=\frac{1}{n!}-\frac{p}{p}\frac{1}{(n-1)!}+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{1}{(n-2)!2!}\cdots\quad+(-)^n\frac{p(p-1)\ldots\{p-(n-1)\}}{p(p-\frac{1}{2})\ldots\{p-\frac{1}{2}(n-1)\}}\frac{1}{n!}$$

$$=\frac{1}{n!}-\frac{p}{2p}\frac{2}{(n-1)!}+\frac{p(p-1)}{2p(2p-1)}\frac{2^2}{(n-2)!2!}\cdots+(-)^n\frac{p(p-1)\ldots\{p-(n-1)\}}{2p(2p-1)\ldots\{2p-(n-1)\}}\frac{2^n}{n!}$$

$$=\frac{1}{2p(2p-1)\ldots\{2p-(n-1)\}}\left\{\frac{2p(2p-1)\ldots\{2p-(n-1)\}}{n!}\right.$$

$$\left.-\frac{(2p-1)(2p-2)\ldots\{2p-(n-1)\}}{(n-1)!}p.2\ldots+(-)^n\frac{p(p-1)\ldots\{p-(n-1)\}}{n!}2^n\right\},$$

and we see that the expression in brackets is equal to the coefficient of t^n in the expansion of

$$(1+t)^{2p}-p.2t(1+t)^{2p-1}+\frac{p(p-1)}{2!}2^2t^2(1+t)^{2p-2}\dots$$
$$+(-)^n\frac{p(p-1)\dots\{p-(n-1)\}}{n!}2^nt^n(1+t)^{2p-n},$$

that is, in

$$(1+t)^{2p}\left(1-\frac{2t}{1+t}\right)^p=(1+t)^p(1-t)^p=(1-t^2)^p.$$

If, therefore, n is uneven the coefficient of a^nx^n in P is zero, and if n is even the coefficient

$$=\frac{1}{2p(2p-1)\dots\{2p-(n-1)\}}\times(-)^{\frac{1}{2}n}\frac{p(p-1)\dots\{p-\frac{1}{2}n+1\}}{(\frac{1}{2}n)!}$$

$$=(-)^{\frac{1}{2}n}\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})\dots\{p-\frac{1}{2}(n-1)\}}\frac{1}{2^n(\frac{1}{2}n)!},$$

which is the coefficient of a^nx^n in U.

Since R differs from P only in having the sign of a changed, and since U is a function of a^2 only, it follows that P=R=U. Also, since Q, S, V differ from P, R, U only in having $-(p+1)$ in place of p, it follows that Q=S=V.

(2°). Suppose p a positive integer, $=i$, say.

In this case the $(i+1)^{\text{th}}$ term of the series in P, including the factor x^{-i}, is

$$x^{-i}.(-)^i\frac{i(i-1)(i-2)\dots\{i-(i-1)\}}{i(i-\frac{1}{2})(i-1)\dots\{i-\frac{1}{2}(i-1)\}}\frac{a^ix^i}{i!},$$

and the next term vanishes owing to the presence of the factor $i-i$ or 0 in the numerator.

For the same reason all the succeeding terms vanish until the factor $i-i$ appears in the denominator also, when the zero factors cancel one another and the series recommences, the first term of the new series being

$$x^{-i}.-\frac{i(i-1)\dots1.0.-1.-2\dots(i-2i)}{i(i-\frac{1}{2})\dots\{i-\frac{1}{2}(2i-1)\}.0}\frac{a^{2i+1}x^{2i+1}}{(2i+1)!}$$

$$=(-)^{i+1}\left(\frac{i!}{2i!}\right)^2\frac{1}{2i+1}\frac{1}{2^{2i}}a^{2i+1}x^{i+1}$$

$$=(-)^{i+1}\frac{2i+1}{\{1.3.5\dots(2i+1)\}^2}a^{2i+1}x^{i+1}$$

$$=gx^{i+1},\text{ where } g=(-)^{i+1}\frac{2i+1}{\{1.3.5\dots(2i+1)\}^2}a^{2i+1}.$$

The new series, multiplied by the factor e^{ax}, thus becomes

$$gx^{i+1}\left\{1-\frac{i+1}{i+1}ax+\frac{(i+1)(i+2)}{(i+1)(i+\frac{3}{2})}\frac{a^2x^2}{2!}-\&c.\right\}e^{ax}$$
$$=gQ.$$

Denoting then by P′ the finite part of P, the series being supposed to end at the term immediately preceding the first term which contains a zero factor in the numerator, viz. putting

$$P'=x^{-i}\left\{1-\frac{i}{i}ax+\frac{i(i-1)}{i(i-\frac{1}{2})}\frac{a^2x^2}{2!}\ldots+(-)^i\frac{i!}{i(i-\frac{1}{2})\ldots\{i-\frac{1}{2}(i-1)\}}\frac{a^ix^i}{i!}\right\}e^{ax},$$

we have found that

$$P=P'+gQ=U.$$

Similarly, if R′ denotes the finite part of R, the series ending at the term immediately preceding the first term which contains a zero factor in the numerator, we find that

$$R=R'-gS=U,$$

and also, as before,

$$Q=S=V.$$

The proof in (1°) that P=U does not apply as it stands when $p=i$, but it can be extended so as to include this case by putting $p=i+h$, and making h indefinitely small. The equality of P and U for all values of p may however be proved without the use of limits by showing that the coefficient of x^n in Ue^{-ax} is equal to the coefficient of x^n in P. To prove this; first suppose n to be even and $=2m$, then the coefficient of $a^{2m}x^{2m}$ in x^pUe^{-ax} is equal to

$$\frac{1}{(2m)!}-\frac{1}{p-\frac{1}{2}}\frac{1}{2^2}\frac{1}{(2m-2)!}+\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\frac{1}{2^4.2!}\frac{1}{(2m-4)!}\ldots$$
$$+(-)^m\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})\ldots(p-m+\frac{1}{2})}\frac{1}{2^{2m}.m!}$$

$$=\frac{1}{2m(2m-1)\ldots(m+1)}\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})\ldots(p-m+\frac{1}{2})}\left\{\frac{(p-\frac{1}{2})(p-\frac{3}{2})\ldots(p-m+\frac{1}{2})}{m!}\right.$$
$$\left.-(m-\tfrac{1}{2})\frac{(p-\frac{3}{2})\ldots(p-m+\frac{1}{2})}{(m-1)!}\ldots+(-)^m\frac{2m(2m-1)\ldots(m+1)}{m!}\frac{1}{2^{2m}}\right\}.$$

The last term

$$=(-)^m\frac{(2m!)}{(m!)^2}\cdot\frac{1}{2^{2m}}=(-)^m\frac{(m-\frac{1}{2})(m-\frac{3}{2})\ldots\frac{1}{2}}{m!},$$

and the expression in brackets is equal to the coefficient of t^m in

$$(1+t)^{p-\frac{1}{2}}-(m-\tfrac{1}{2})t(1+t)^{p-\frac{3}{2}}+\frac{(m-\frac{1}{2})(m-\frac{3}{2})}{2!}t^2(1+t)^{p-\frac{5}{2}}\ldots+(-)^m\frac{(m-\frac{1}{2})\ldots\frac{1}{2}}{m!}t^m(1+t)^{p-m-\frac{1}{2}}$$

$$=(1+t)^{p-\frac{1}{2}}\left\{1-\frac{t}{1+t}\right\}^{m-\frac{1}{2}} \qquad =(1+t)^{p-m}.$$

The coefficient of t^m in the expansion of $(1+t)^{p-m}$ is equal to

$$\frac{(p-m)(p-m-1)\ldots(p-2m+1)}{m!},$$

and therefore the coefficient of $a^{2m}x^{2m}$ in $x^p.\mathrm{U}e^{-ax}$ is equal to

$$\frac{1}{(2m)!}\,\frac{(p-m)(p-m-1)\ldots(p-2m+1)}{(p-\frac{1}{2})(p-\frac{3}{2})\ldots(p-m+\frac{1}{2})},$$

which is the coefficient of $a^{2m}x^{2m}$ in $x^p.\mathrm{P}$ when the factors $p(p-1)(p-2)\ldots(p-m+1)$ are divided out from the numerator and denominator.

Similarly, if $n=2m+1$, the coefficient of $a^{2m+1}x^{2m+1}$ in $x^p.\mathrm{U}e^{-ax}$ is found to be equal to

$$-\frac{1}{(2m+1)!}\,\frac{(p-m-1)(p-m-2)\ldots(p-2m)}{(p-\frac{1}{2})(p-\frac{3}{2})\ldots(p-m+\frac{1}{2})},$$

which is the coefficient of $a^{2m+1}x^{2m+1}$ in $x^p.\mathrm{P}$ when the factors $p(p-1)\ldots(p-m)$ are divided out from the numerator and denominator.

Thus, if $p=i$, the coefficients of the terms involving x^{i+1}, x^{i+2}, ... x^{2i} in the series in P vanish, and we have $\mathrm{U}=\mathrm{P}'+g\mathrm{Q}$.

(3°) If p = a negative integer $=-i-1$, then Q and S involve zero terms, and, denoting by Q′ and S′ the values of Q and S when the series are supposed to terminate at the term preceding the first term involving a zero factor in the numerator, V, Q, and S become equal to U, P, and R when p is put equal to i, that is, to the U, P, and R of (2°) and *vice versâ*. In this case, therefore,

$$\mathrm{Q}=\mathrm{Q}'+g\mathrm{P}=\mathrm{V}=\mathrm{S}=\mathrm{S}'-g\mathrm{R},$$

and

$$\mathrm{P}=\mathrm{R}=\mathrm{U}.$$

The relations between the particular integrals in the three cases are therefore

(1°) p not = an integer,

$$\mathrm{P}=\mathrm{R}=\mathrm{U}, \qquad \mathrm{Q}=\mathrm{S}=\mathrm{V}.$$

(2°) p = a positive integer,

$$P=R=U=\tfrac{1}{2}(P'+R'), \qquad Q=S=V=\frac{1}{2g}(R'-P');$$

(3°) p = a negative integer,

$$P=R=U=\frac{1}{2g}(S'-Q'), \qquad Q=S=V=\tfrac{1}{2}(Q'+S');$$

6. If we suppose the series always to terminate directly a zero factor appears in a numerator (so that P', Q', R', S' are now denoted by P, Q, R, S), the relations are

(1°) p not = an integer,

$$P=R=U, \qquad Q=S=V;$$

(2°) p = a positive integer,

$$Q=S=V=\frac{1}{2g}(R-P), \qquad U=\tfrac{1}{2}(P+R);$$

(3°) p = a negative integer,

$$P=R=U=\frac{1}{2g}(S-Q), \qquad V=\tfrac{1}{2}(Q+S);$$

The change of form of the relations, which in this mode of statement appears so remarkable, does not, as we have seen, occur if the series be supposed to extend to infinity in all cases.

It may be observed that it is clear from the manner in which the series were obtained in arts. 1 and 2 that we are always at liberty to stop at the term immediately preceding the first term containing a zero factor in the numerator, as this finite portion of the series satisfies the differential equation, and that the second series obtained by allowing the terms to recommence and to extend to infinity also satisfies the differential equation.

The phrase "term preceding the first term containing a zero factor in the numerator" has been used in preference to "term preceding the first zero term" in order to include the cases of $p=0$ or $p=-1$, in which no zero term occurs.

7. It was shown in art. 5 that

$$\left(1-\frac{p}{p}ax+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{a^2x^2}{2!}-\frac{p(p-1)(p-2)}{p(p-\frac{1}{2})(p-1)}\frac{a^3x^3}{3!}+\&c.\right)e^{ax}$$

$$=1-\frac{1}{p-\frac{1}{2}}\frac{a^2x^2}{2^2}+\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\frac{a^4x^4}{2^4.2!}-\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})(p-\frac{5}{2})}\frac{a^6x^6}{2^6.3!}+\&c.$$

Putting $2p=-m-1$ in this identity, we have

$$\left(1-\frac{m+1}{m+1}ax+\frac{(m+1)(m+3)}{(m+1)(m+2)}\frac{a^2x^2}{2!}-\frac{(m+1)(m+3)(m+5)}{(m+1)(m+2)(m+3)}\frac{a^3x^3}{3!}+\&c.\right)e^{ax}$$

$$=1+\frac{1}{m+2}\frac{a^2x^2}{2}+\frac{1}{(m+2)(m+4)}\frac{a^4x^4}{2^2.2!}+\frac{1}{(m+2)(m+4)(m+6)}\frac{a^6x^6}{2^3.3!}+\&c.$$

The right-hand side of this equation is unaltered by a change of sign of x, and therefore, putting $a=1$,

$$\left(1-\frac{m+1}{m+1}x+\frac{(m+1)(m+3)}{(m+1)(m+2)}\frac{x^2}{2!}-\frac{(m+1)(m+3)(m+5)}{(m+1)(m+2)(m+3)}\frac{x^3}{3!}+\&c.\right)e^{x}$$

$$=\left(1+\frac{m+1}{m+1}x+\frac{(m+1)(m+3)}{(m+1)(m+2)}\frac{x^2}{2!}+\frac{(m+1)(m+3)(m+5)}{(m+1)(m+2)(m+3)}\frac{x^3}{3!}+\&c.\right)e^{-x}.$$

which is true for all values of m, except m = a negative even integer.

Writing n in place of $m+1$, it follows that

$$e^{2x}=\frac{1+x+\frac{n+2}{n+1}\frac{x^2}{2!}+\frac{(n+2)(n+4)}{(n+1)(n+2)}\frac{x^3}{3!}+\&c.}{1-x+\frac{n+2}{n+1}\frac{x^2}{2!}-\frac{(n+2)(n+4)}{(n+1)(n+2)}\frac{x^3}{3}+\&c.},$$

which is true for all values of n, except n = a negative uneven integer. Several deductions from this formula are given in a paper "Generalised Form of Certain Series" ('Proceedings of the London Mathematical Society,' vol. ix., pp. 197–204, 1878).

§ II.

Integration of the differential equation when p = an integer. Arts. 8–15.

8. A particular integral of the equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{h^2}{x^2}\frac{d^2u}{dh^2}$$

is

$$u=e^{a\sqrt{(x^2+xh)}},$$

for, from this value of u we find at once by differentiation

$$\frac{d^2u}{dx^2}=a^2u\frac{(x+\frac{1}{2}h)^2}{x^2+xh}-au\frac{\frac{1}{4}h^2}{(x^2+xh)^{\frac{3}{2}}},$$

$$\frac{d^2u}{dh^2}=a^2u\frac{\frac{1}{4}x^2}{x^2+xh}\quad-au\frac{\frac{1}{4}x^2}{(x^2+xh)^{\frac{3}{2}}},$$

whence

$$\frac{d^2u}{dx^2}-a^2u=\frac{h^2}{x^2}\frac{d^2u}{dh^2}.$$

9. Let the above value of u be expanded in powers of h, so that

$$u=e^{a\sqrt{(x^2+xh)}}=P_0+P_1h+P_2h^2\ldots+P_ih^i+P_{i+1}h^{i+1}+\&c.,$$

then

$$\frac{d^2u}{dx^2}-a^2u=\qquad\ldots+\left(\frac{d^2P_{i+1}}{dx^2}-a^2P_{i+1}\right)h^{i+1}+\&c.,$$

$$\frac{h^2}{x^2}\frac{d^2u}{dh^2}=\qquad\ldots+\frac{(i+1)i}{x^2}h^{i+1}+\&c.,$$

and therefore P_{i+1} satisfies the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{i(i+1)}{x^2}u.$$

Thus the general integral of this differential equation is

$$\begin{aligned}u=&\ \text{A. coefficient of } h^{i+1} \text{ in expansion of } e^{a\sqrt{(x^2+xh)}}\\ &+\text{B. coefficient of } h^{i+1} \text{ in expansion of } e^{-a\sqrt{(x^2+xh)}}\end{aligned}$$

The particular integrals to which the different modes of expansion of $e^{a\sqrt{(x^2+xh)}}$ lead will now be examined, and connected with the forms already obtained in § I.

10. The coefficient of h^{i+1} in the expansion of $e^{a\sqrt{(x^2+xh)}}$ is equal to the coefficient of h^{i+1} in

$$1+a(x^2+xh)^{\frac{1}{2}}+\frac{a^2}{2!}(x^2+xh)+\frac{a^3}{3!}(x^2+xh)^{\frac{3}{2}}+\frac{a^4}{4!}(x^2+xh)^2+\frac{a^5}{5!}(x^2+xh)^{\frac{5}{2}}+\&c.,$$

and the coefficient of h^{i+1} in $(x^2+xh)^{\frac{1}{2}(2n-1)}$

$$=\frac{(n-\frac{1}{2})(n-\frac{3}{2})\ldots(n-i-\frac{1}{2})}{(i+1)!}x^{2n-i-2}.$$

Thus the coefficient of h^{i+1} in the terms involving uneven powers of a

$$=\frac{1}{2}\frac{(-1)^i}{(i+1)!}\frac{1}{2}\cdot\frac{3}{2}\ldots(i-\frac{1}{2})x^{-i}\left\{a-\frac{a^3}{3!}\frac{\frac{3}{2}}{i-\frac{1}{2}}x^2+\frac{a^5}{5!}\frac{\frac{3}{2}\cdot\frac{5}{2}}{(i-\frac{1}{2})(i-\frac{3}{2})}x^4-\&c.\right\}$$

$$=\frac{1}{2}a\frac{(-1)^i}{(i+1)!}\frac{1}{2}\cdot\frac{3}{2}\ldots(i-\frac{1}{2})x^{-i}\left\{1-\frac{1}{i-\frac{1}{2}}\frac{a^2x^2}{2^2}+\frac{1}{(i-\frac{1}{2})(1-\frac{3}{2})}\frac{a^4x^4}{2^4 2!}-\&c.\right\}=\lambda U,$$

where

$$\lambda=\tfrac{1}{2}a\frac{(-1)^i}{(i+1)!}\tfrac{1}{2}\cdot\tfrac{3}{2}\ldots(i-\tfrac{1}{2})=(-)^i\frac{1.3.5\ldots(2i-1)}{2.4.6\ldots(2i+2)}a.$$

Of the terms involving even powers of a the first that contains a term in h^{i+1} is

$$\frac{a^{2i+2}}{(2i+2)!}x^{i+1}(x+h)^{i+1},$$

so that the coefficient of h^{i+1} in the terms involving even powers of a

$$=\frac{a^{2i+2}}{(2i+2)!}x^{i+1}+\frac{a^{2i+4}}{(2i+4)!}(i+2)x^{i+3}+\frac{a^{2i+6}}{(2i+6)!}\frac{(i+2)(i+3)}{2!}x^{i+5}+\&c.$$

$$=\frac{a^{2i+2}}{(2i+2)!}x^{i+1}\left\{1+\frac{1}{i+\frac{3}{2}}\frac{a^2x^2}{2!}+\frac{1}{(i+\frac{3}{2})(i+\frac{5}{2})}\frac{a^4x^4}{2^4 2!}+\&c.\right\}=\frac{a^{2i+2}}{(2i+2)!}\mathrm{V}.$$

The complete coefficient of h^{i+1} in the expansion of $e^{a\sqrt{(x^2+xh)}}$ therefore

$$=\lambda\mathrm{U}+\frac{a^{2i+2}}{(2i+2)!}\mathrm{V}=\lambda\left\{\mathrm{U}+(-)^i a^{2i+1}\frac{2i+1}{(1.3.5\ldots 2i+1)^2}\mathrm{V}\right\}$$

$$=\lambda\{\mathrm{U}-g\mathrm{V}\},$$

g being the same as in art 5.

11. Now

$$e^{a\sqrt{(x^2+xh)}}=e^{ax}.\,e^{a\{\sqrt{(x^2+xh)}-x\}}=e^{-ax}.\,e^{a\{\sqrt{(x^2+xh)}+x\}},$$

and we obtain other forms of the integral by finding the coefficients of h^{i+1} in the expansion of $e^{a\{\sqrt{(x^2+xh)}-x\}}$ and of $e^{a\{\sqrt{(x^2+xh)}+x\}}$, and multiplying them by e^{ax} and e^{-ax} respectively.

It is well known that

$$\left\{1-\sqrt{(1-4t)}\right\}^n=2^nt^n\left\{1+nt+\frac{n(n+3)}{2!}t^2+\frac{n(n+4)(n+5)}{3!}t^3+\&c.\right\},$$

and

$$\left\{1+\sqrt{(1-4t)}\right\}^n=2^n\left\{1-nt+\frac{n(n-3)}{2!}t^2-\frac{n(n-4)(n-5)}{3!}t^3+\&c.\right\},$$

where in the second series, if n is an even positive integer the coefficients of the $\frac{1}{2}n-1$ terms involving $t^{\frac{1}{2}n+1}$, $t^{\frac{1}{2}n+2}\ldots t^{n-1}$ are zero, and if n is an uneven positive integer the coefficients of the $\frac{1}{2}(n-1)$ terms involving $t^{\frac{1}{2}(n+1)}$, $t^{\frac{1}{2}(n+3)}\ldots t^{n-1}$ are zero.

Putting $t=-\frac{h}{4x}$, these formulæ become

$$\{\sqrt{(x^2+xh)}-x\}^n=\frac{1}{2^n}h^n\left\{1-n\frac{h}{4x}+\frac{n(n+3)}{2!}\frac{h^2}{4^2x^2}-\frac{n(n+4)(n+5)}{3!}\frac{h^3}{4^3x^3}+\&c.\right\},$$

$$\{\sqrt{(x^2+xh)}+x\}^n=2^nx^n\left\{1+n\frac{h}{4x}+\frac{n(n-3)}{2!}\frac{h^2}{4^2x^2}+\frac{n(n-4)(n-5)}{3!}\frac{h^3}{4^3x^3}+\&c.\right\}.$$

The coefficient of h^{i+1} in $\{\sqrt{(x^2+xh)}-x\}^n$ therefore

$$=\frac{1}{2^n}(-)^{i+1-n}\frac{n(i+2)(i+3)\dots(2i+1-n)}{(i+1-n)!}\frac{1}{4^{i+1-n}x^{i+1-n}}=\frac{(-1)^{i+1}}{4^{i+1}x^{i+1}}\cdot(-)^n\frac{(i+2)\dots(2i+1-n)}{(i+1-n)!}n2^nx^n,$$

and the coefficient of h^{i+1} in $\{\sqrt{(x^2+xh)}+x\}^n$

$$=2^nx^n\frac{n(n-i-2)(n-i-3)\dots(n-2i-1)}{(i+1)!}\frac{1}{4^{i+1}x^{i+1}}=\frac{(-1)^i}{4^{i+1}x^{i+1}}\cdot\frac{(i+2-n)\dots(2i+1-n)}{(i+1)!}n2^nx^n.$$

12. The coefficient of h^{i+1} in $e^{a\{\sqrt{(x^2+xh)}-x\}}$, that is in

$$1+a\{\sqrt{(x^2+xh)}-x\}+\frac{a^2}{2!}\{\sqrt{(x^2+xh)}-x\}^2+\frac{a^3}{3!}\{\sqrt{(x^2+xh)}-x\}^3+\&c.,$$

is, by the last article, equal to

$$\frac{(-1)^i}{4^{i+1}x^{i+1}}\Big\{a\frac{(i+2)\dots 2i}{i!}2x-\frac{a^2}{2!}\frac{(i+2)\dots(2i-1)}{(i-1)!}2.2^2x^2$$
$$+\frac{a^3}{3!}\frac{(i+2)\dots(2i-3)}{(i-2)!}3.2^3x^3-\dots+(-)^i\frac{a^{i+1}}{(i+1)!}2^{i+1}x^{i+1}\Big\},$$

for, when n is greater than $i+1$, there is no term involving h^{i+1}.

This expression

$$=(-)^i\tfrac{1}{2}a\frac{(i+2)\dots 2i}{4^i.i!}\frac{1}{x^i}\Big\{1-a\frac{i}{2i}2x+\frac{a^2}{2!}\frac{i(i-1)}{2i(2i-1)}2^2x^2\dots+(-)^i\frac{a^i}{i!}\frac{i!}{2i(2i-1)\dots(i+1)}2^ix^i\Big\},$$

$$=\lambda\frac{1}{x^i}\Big\{1-\frac{i}{i}ax+\frac{i(i-1)}{i(i-\frac{1}{2})}\frac{a^2x^2}{2!}\dots+(-)^i\frac{i(i-1)\dots\{i-(i-1)\}}{i(i-\frac{1}{2})\dots\{i-\frac{1}{2}(i-1)\}}\frac{a^ix^i}{i!}\Big\},$$

for the constant multiplier

$$=(-)^i\tfrac{1}{2}a\frac{(i+2)\dots 2i}{4^i.i!}=(-)^i\frac{(2i)!}{(2.4.6\dots 2i)^2(2i+2)}a=(-)^i\frac{1.3.5\dots(2i-1)}{2.4.6\dots(2i+2)}a,$$

which is the quantity denoted by λ in art. 10.

The coefficient of h^{i+1} in the expansion of $e^{ax}.e^{a\{\sqrt{(x^2+xh)}-x\}}$ is therefore $=\lambda P'$.

13. The coefficient of h^{i+1} in $e^{a\{\sqrt{(x^2+xh)}+x\}}$, that is in

$$1+a\{\sqrt{(x^2+xh)}+x\}+\frac{a^2}{2!}\{\sqrt{(x^2+xh)}+x\}^2+\frac{a^3}{3!}\{\sqrt{(x^2+xh)}+x\}^3+\&c.,$$

is, by art. 10, equal to

$$\frac{(-1)^i}{(i+1)!}\frac{1}{4^{i+1}x^{i+1}}\Big[a\{(i+1)\ldots 2i\}2x+\frac{a^2}{2!}\{i\ldots(2i-1)\}2.2^2x^2\ldots$$

$$+\frac{a^{i+1}}{(i+1)!}\{1.2\ldots i\}(i+1)2^{i+1}x^{i+1}$$

$$+\frac{a^{2i+2}}{(2i+2)!}\{(-i)(-i+1)\ldots(-1)\}(2i+2)2^{2i+2}x^{2i+2}$$

$$+\frac{a^{2i+3}}{(2i+3)!}\{(-i-1)\ldots(-2)\}(2i+3)2^{2i+3}x^{2i+3}+\&c.\Big]$$

$$=\lambda\frac{1}{x^i}\left\{1+\frac{i}{i}ax+\frac{i(i-1)}{i(i-\frac{1}{2})}\frac{a^2x^2}{2!}\ldots+\frac{i(i-1)\ldots\{i-(i-1)\}}{i(i-\frac{1}{2})\ldots\{i-\frac{1}{2}(i-1)\}}\frac{a^ix^i}{i!}\right\}$$

$$+(-)^i\frac{i!}{(2i+1)!}a^{2i+2}x^{i+1}\left\{1+\frac{i+1}{i+1}ax+\frac{(i+1)(i+2)}{(i+1)(i+\frac{3}{2})}\frac{a^2x^2}{2!}+\&c.\right\}.$$

The coefficient of h^{i+1} in the expansion of $e^{-ax}.e^{a\{\surd(x^2+xh)+x\}}$ is therefore

$$\lambda R'+\frac{a^{2i+2}}{(i+1).(2i+1)!}S;$$

and we have

$$\frac{1}{\lambda}\frac{a^{2i+2}}{(i+1).(2i+1)!}=(-)^i\frac{2(2i+1)}{\{1.3.5\ldots(2i+1)\}^2}a^{2i+1}=-2g,$$

so that the coefficient of h^{i+1}

$$=\lambda(R'-2gS).$$

14. Thus the three forms of the same integral which are obtained by the expansion of

$$e^{a\surd(x^2+xh)},\qquad e^{ax}.e^{a\{\surd(x^2+xh)-x\}},\qquad e^{-ax}.e^{a\{\surd(x^2+xh)+x\}}$$

are

$$U-gV,\qquad P',\qquad R'-2gS.$$

Changing the sign of a, we obtain as the coefficient of h^{i+1} in the expansion of

$$e^{-a\surd(x^2+xh)},\qquad e^{-ax}.e^{-a\{\surd(x^2+xh)-x\}},\qquad e^{ax}.e^{-a\{\surd(x^2+xh)+x\}}$$

the values $-\lambda(U+gV)$, $-\lambda R'$, $-\lambda(P'+2gQ)$, giving the three equal integrals

$$U+gV,\qquad R',\qquad P'+2gQ.$$

Therefore

$$U-gV=P'=R'-2gS,$$

$$U+gV=R'=P'+2gQ.$$

whence

$$U=\tfrac{1}{2}(P'+R'), \qquad Q=S=V=\frac{1}{2g}(R'-P'),$$

which agree with the relations found for the case of $p=$ a positive integer in art. 5.

If p is a negative integer $=-i-1$, then $p(p+1)=i(i+1)$; we may therefore replace i by $-i-1$ throughout in the integrals just obtained, and thus deduce the system of integrals considered in (3°) of art. 5.

15. It may be observed that, since the series for $\{1-\sqrt{(1-4t)}\}^n$ and $\{1+\sqrt{(1-4t)}\}^n$ in art. 11 terminate and recommence when n is respectively a negative or positive integer, it is evident that the solutions in series of the differential equation satisfied by them will present points of similarity to the solutions Q and P of (1). The former differential equation is

$$t(1-4t)\frac{d^2u}{dt^2}+\{(4n-6)t-n+1\}\frac{du}{dt}-n(n-1)u=0,$$

and its integration in series is considered in a paper "Example Illustrative of a Point in the Solution of Differential Equations in Series" ('Messenger of Mathematics,' vol. viii., pp. 20–23).

§ III.

Transformations of the original differential equation. RICCATI'S *equation.* Arts. 16, 17.

16. If the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p+1)}{x^2}u \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is transformed by assuming $u=x^{-p}v$, it becomes

$$\frac{d^2v}{dx^2}-\frac{2p}{x}\frac{dv}{dx}-a^2v=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

This equation therefore admits of integration in a finite form when $p=$ an integer, and the six particular integrals U_1, V_1, P_1, Q_1, R_1, S_1, which are equal respectively to x^pU, x^pV, x^pP, x^pQ, x^pR, x^pS are connected with one another, in the different cases, by the same relations as those found for U, V, P, Q, R, S in art. 5.

If we put $2p=n-1$, so that the differential equation becomes

$$\frac{d^2v}{dx^2}-\frac{n-1}{x}\frac{dv}{dx}-a^2v=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

then the six integrals take the forms

$$U_1= \quad 1-\frac{1}{n-2}\frac{a^2x^2}{2}+\frac{1}{(n-2)(n-4)}\frac{a^4x^4}{2^2.2!}-\frac{1}{(n-2)(n-4)(n-6)}\frac{a^6x^6}{2^3.3!}+\&c.,$$

$$V_1=x^n\left\{1+\frac{1}{n+2}\frac{a^2x^2}{2}+\frac{1}{(n+2)(n+4)}\frac{a^4x^4}{2^2.2!}+\frac{1}{(n+2)(n+4)(n+6)}\frac{a^6x^6}{2^3.3!}+\&c.\right\},$$

$$P_1= \quad \left\{1-\frac{n-1}{n-1}ax+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{a^2x^2}{2!}-\frac{(n-1)(n-3)(n-5)}{(n-1)(n-2)(n-3)}\frac{a^3x^3}{3!}+\&c.\right\}e^{ax},$$

$$Q_1=x^n\left\{1-\frac{n+1}{n+1}ax+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{a^2x^2}{2!} \quad \frac{(n+1)(n+3)(n+5)}{(n+1)(n+2)(n+3)}\frac{a^3x^3}{3!}+\&c.\right\}e^{ax},$$

$$R_1= \quad \left\{1+\frac{n-1}{n-1}ax+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{a^2x^2}{2!}+\frac{(n-1)(n-3)(n-5)}{(n-1)(n-2)(n-3)}\frac{a^3x^3}{3!}+\&c.\right\}e^{-ax},$$

$$S_1=x^n\left\{1+\frac{n+1}{n+1}ax+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{a^2x^2}{2!}+\frac{(n+1)(n+3)(n+5)}{(n+1)(n+2)(n+3)}\frac{a^3x^3}{3!}+\&c.\right\}e^{-ax}.$$

The differential equation admits of integration in a finite form if $n=$ an uneven integer, and the relations between the particular integrals are the same as in art. 5, viz., accented letters denoting the terminated series as before,

(1°.) n not $=$ an integer,

$$P_1=R_1=U_1, \qquad Q_1=S_1=V_1\ ;$$

(2°.) $n=$ a positive uneven integer,

$$P_1=R_1=U_1=\tfrac{1}{2}(P_1'+R_1'), \qquad Q_1=S_1=V_1=\frac{1}{2g_1}(R_1'-P_1')\ ;$$

(3°.) $n=$ a negative uneven integer,

$$P_1=R_1=U_1=\frac{1}{2g_1}(S_1'-Q_1'), \qquad Q_1=S_1=V_1=\tfrac{1}{2}(Q_1'+S_1');$$

where

$$g_1=(-)^{\frac{1}{2}(n+1)}\frac{n}{1^2.3^2.5^2\ldots n^2}a^n.$$

This is perhaps the simplest form in which the six integrals can be exhibited; and, having regard merely to the simplicity of the series and to the expression of the manner in which they are related to one another, (3) should be preferred as the standard form of the differential equation both to the original form (1) and to RICCATI's equation (4), which is considered in the next article. The form (3) is that adopted by BACH in his memoir (see iv. of § VIII.).

It may be observed that if $p=i$, a positive integer, the differential equation (2) is satisfied by $v=\frac{1}{x}\times$ coefficient of h^{i+1} in the expansion of $e^{ax\sqrt{(1+h)}}$, and if $p=-i$, by $v=x\times$ coefficient of h^{i+1} in the expansion of $e^{a\sqrt{(x^2+h)}}$; these results follow from § II.

17. Transforming the equation (3) by assuming $x=nz^{\frac{1}{n}}$, it becomes

$$z^{2-\frac{2}{n}}\frac{d^2v}{dz^2}-a^2v=0,$$

or, putting $n=\frac{1}{q}$,

$$\frac{d^2v}{dz^2}-a^2z^{2q-2}v=0 \qquad \text{(4)}.$$

RICCATI's equation in its original form is

$$\frac{dy}{dz}+by^2=cz^m;$$

it may without loss of generality be written

$$\frac{dy}{dz}+y^2=z^m,$$

and, putting $y=\frac{1}{v}\frac{dv}{dz}$, it becomes

$$\frac{d^2v}{dz^2}-z^mv=0.$$

Thus (4) is the equation derived from

$$\frac{dy}{dz}+y^2=a^2z^{2q-2}$$

by assuming $y=\frac{1}{v}\frac{dv}{dz}$, and it is convenient to regard it as the standard form of RICCATI's equation.

The six particular integrals of (4) are

$$U_2= 1+\frac{a^2z^{2q}}{2q(2q-1)}+\frac{a^4z^{4q}}{2q(2q-1)4q(4q-1)}+\frac{a^6z^6}{2q(2q-1)4q(4q-1)6q(6q-1)}+\&c.,$$

$$V_2=z\left\{1+\frac{a^2z^{2q}}{2q(2q+1)}+\frac{a^4z^{4q}}{2q(2q+1)4q(4q+1)}+\frac{a^6z^6}{2q(2q+1)4q(4q+1)6q(6q+1)}+\&c.\right\},$$

$$P_2=\left\{1-\frac{q-1}{q(q-1)}az^q+\frac{(q-1)(3q-1)}{q(q-1)2q(2q-1)}a^2z^{2q}-\frac{(q-1)(3q-1)(5q-1)}{q(q-1)2q(2q-1)3q(3q-1)}a^3z^{3q}+\&c.\right\}e^{\frac{a}{q}z^q},$$

$$Q_2=z\left\{1-\frac{q+1}{q(q+1)}az^q+\frac{(q+1)(3q+1)}{q(q+1)2q(2q+1)}a^2z^{2q}-\frac{(q+1)(3q+1)(5q+1)}{q(q+1)2q(2q+1)3q(3q+1)}a^3z^{3q}+\&c.\right\}e^{\frac{a}{q}z^q},$$

$$R_2=\left\{1+\frac{q-1}{q(q-1)}az^q+\frac{(q-1)(3q-1)}{q(q-1)2q(2q-1)}a^2z^{2q}+\frac{(q-1)(3q-1)(5q-1)}{q(q-1)2q(2q-1)3q(3q-1)}a^3z^{3q}+\&c.\right\}e^{-\frac{a}{q}z^q},$$

$$S_2=z\left\{1+\frac{q+1}{q(q+1)}az^q+\frac{(q+1)(3q+1)}{q(q+1)2q(2q+1)}a^2z^{2q}+\frac{(q+1)(3q+1)(5q+1)}{q(q+1)2q(2q+1)3q(3q+1)}a^3z^{3q}+\&c.\right\}e^{-\frac{a}{q}z^q}.$$

The differential equation admits of integration in a finite form if q = the reciprocal of an uneven integer, and, the terminated series being denoted by accented letters as before, the relations between the particular integrals are the same as in art. 5, viz.

(1°) q not = the reciprocal of an uneven integer,

$$P_2=R_2=U_2, \qquad Q_2=S_2=V_2\,;$$

(2°) q = the reciprocal of an uneven positive integer,

$$P_2=R_2=U_2=\tfrac{1}{2}(P_2'+R_2'), \quad Q_2=S_2=V_2=\left(\frac{1}{q}\right)^{1-\frac{1}{q}}\frac{1}{2g_2}(R_2'-P_2')\,;$$

(3°) q = the reciprocal of an uneven negative integer,

$$P_2=R_2=U_2=\left(\frac{1}{q}\right)^{1+\frac{1}{q}}\frac{1}{2g_2}(S_2'-Q_2'), \quad Q_2=S_2=V_2=\tfrac{1}{2}(Q_2'+S_2')\,;$$

where

$$g_2=\frac{(-1)^{\frac{1}{2}\left(1+\frac{1}{q}\right)}}{1^2.3^2.5^2\ldots\frac{1}{q^2}}a^{\frac{1}{q}}.$$

The integrals P_2, Q_2, R_2, S_2 were given by CAYLEY in the 'Philosophical Magazine,' Fourth series, vol. 36, pp. 348–351 (November, 1868).

§ IV.

Special forms of the particular integrals in the cases in which the differential equations admit of integration in a finite form. Arts. 18, 19.

18. When the differential equations admit of integration in series containing a finite number of terms, these finite particular integrals may be presented in another form by commencing the terminating series at the other end.

Thus in the case of the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p+1)}{x^2}u,$$

if p is a positive integer, the particular integral

$$\frac{1}{x^p}\left\{1-\frac{p}{p}ax+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{a^2x^2}{2!}\dots+(-)^{p-1}\frac{p(p-1)\dots 2}{p(p-\frac{1}{2})\dots\frac{1}{2}(p+2)}\frac{a^{p-1}x^{p-1}}{(p-1)!}\right.$$
$$\left.+(-)^p\frac{p(p-1)\dots 1}{p(p-\frac{1}{2})\dots\frac{1}{2}(p+1)}\frac{a^px^p}{p!}\right\}e^{ax}$$

$$=(-)^p\frac{2^pa^p}{(p+1)\dots 2p}\left\{1-\frac{p}{1}\cdot\frac{1}{2}(p+1)\frac{1}{ax}+\frac{p(p-1)}{1.2}\frac{1}{2}(p+1)\frac{1}{2}(p+2)\frac{1}{a^2x^2}\dots\right.$$
$$\left.+(-)^p\frac{1}{2}(p+1)\frac{1}{2}(p+2)\dots\frac{1}{2}(2p)\frac{1}{a^px^p}\right\}e^{ax}$$

$$=(-)^p\frac{2^pa^p}{(p+1)\dots 2p}\left\{1-\frac{p(p+1)}{2}\frac{1}{ax}+\frac{(p-1)p(p+1)(p+2)}{2.4}\frac{1}{a^2x^2}\dots\right.$$
$$\left.+(-)^p\frac{1.2\dots 2p}{2.4\dots 2p}\frac{1}{a^px^p}\right\}e^{ax};$$

so that, if p = an integer, the finite particular integrals are

$$\left\{1-\frac{p(p+1)}{2}\frac{1}{ax}+\frac{(p-1)p(p+1)(p+2)}{2.4}\frac{1}{a^2x^2}-\text{\&c.}\right\}e^{ax}$$

(the series being continued till it terminates of itself through the terms all containing a zero factor), and a similar expression derived from this by changing the sign of a.

19. Similarly, if n = an uneven integer, then

$$x^{\frac{1}{2}(n-1)}\left\{1-(n^2-1^2)\left(\frac{1}{8ax}\right)+\frac{(n^2-1^2)(n^2-3^2)}{1.2}\left(\frac{1}{8ax}\right)^2-\frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{1.2.3}\left(\frac{1}{8ax}\right)^3+\text{\&c.}\right\}e^{ax}$$

and a similar expression derived from this by changing the sign of a, are particular integrals of the differential equation

$$\frac{d^2v}{dx^2}-\frac{n-1}{x}\frac{dv}{dx}-a^2v=0.$$

In the case of RICCATI's equation,

$$\frac{d^2v}{dz^2}-a^2z^{2q-2}v=0,$$

if $q=$ the reciprocal of an uneven integer, the two particular integrals are

$$z^{\frac{1}{2}(1-q)}\left\{1+\frac{q^2-1}{q}\left(\frac{1}{8az^q}\right)+\frac{(q^2-1)(3^2q^2-1)}{q.2q}\left(\frac{1}{8az^q}\right)^2+\frac{(q^2-1)(3^2q^2-1)(5^2q^2-1)}{q.2q.3q}\left(\frac{1}{8az^q}\right)^3+\&c.\right\}e^{\frac{a}{q}z^q}$$

and a similar expression derived from this by changing the sign of a.

These appear to be the best forms in which the integrals can be presented when the equations admit of solution in a finite form: but they do not suggest the solutions for the general cases when the letters are unrestricted. The series ultimately become divergent when they do not terminate.

§ V.

Evaluation of definite integrals satisfying the differential equations. Arts. 20–28.

20. It was shown by POISSON* that the definite integral

$$y=\int_0^\infty e^{-x^m-\frac{bz^m}{x^m}}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

satisfies the RICCATI's equation

$$\frac{d^2y}{dz^2}=m^2bz^{m-2}y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

so that the value of the integral must be of the form AU_2+BV_2, where U_2, V_2 are the same as in art. 17, and $a^2=m^2b$, $q=\frac{1}{2}m$; it remains to determine the constants A and B.

* 'Journal de l'École Polytechnique,' Cahier xvi. (vol. ix., 1813), p. 237.

It is however more convenient in the first place to consider the definite integral in the form

$$y=\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}\,dx,$$

which is obtained by transforming (5) by the substitution $x^m=x'^2$, for the integral (5) thus becomes

$$\int_0^\infty \frac{2}{m}\,x^{\frac{2}{m}-1}e^{-x^2-\frac{bz^m}{x^2}}\,dx.$$

Comparing (6) with the standard form (4) of RICCATI's equation in art. 17, we have $m=2q$, so that $\frac{2}{m}=\frac{1}{q}=n$, and $m^2b=a^2$.

Let $bz^m=\alpha^2$; then $z=b^{-\frac{1}{2}n}\alpha^n$, and we see that the definite integral

$$y=\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}\,dx$$

satisfies the differential equation

$$\frac{d^2y}{d\alpha^2}-\frac{n-1}{\alpha}\frac{dy}{d\alpha}-4y=0.$$

The value of this definite integral is therefore of the form AU_1+BV_1, where U_1, V_1 are the same as in art. 16, α being substituted for x and a put $=2$: viz., writing

$$M=1-\frac{1}{n-2}(2\alpha^2)+\frac{1}{(n-2)(n-4)}\frac{(2\alpha^2)^2}{2!}-\frac{1}{(n-2)(n-4)(n-6)}\frac{(2\alpha^2)^3}{3!}+\&c.,$$

$$N=1+\frac{1}{n+2}(2\alpha^2)+\frac{1}{(n+2)(n+4)}\frac{(2\alpha^2)^2}{2!}+\frac{1}{(n+2)(n+4)(n+6)}\frac{(2\alpha^2)^3}{3!}+\&c.,$$

then

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}\,dx=AM+B\alpha^nN.$$

Suppose n positive, and put $\alpha=0$; we thus find

$$A=\int_0^\infty x^{n-1}e^{-x^2}dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n),$$

and therefore

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}\,dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)M+\alpha^n\phi(n)N.$$

Transform the integral by assuming $x=\frac{\alpha}{x'}$; this equation then becomes

$$\int_0^\infty x^{-n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\alpha^{-n}\mathrm{M}+\phi(n)\mathrm{N}.$$

whence, changing the sign of n,

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx=\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\alpha^n\mathrm{N}+\phi(-n)\mathrm{M};$$

and it follows therefore that $\phi(n)=\frac{1}{2}\Gamma(-\frac{1}{2}n)$.

Thus for all values of n (except, of course, $n=$ an even integer)

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\mathrm{M}+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\alpha^n\mathrm{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$=\quad \tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\left\{1-\frac{n-1}{n-1}(2\alpha)+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{(2\alpha)^2}{2!}-\frac{(n-1)(n-3)(n-5)}{(n-1)(n-2)(n-3)}\frac{(2\alpha)^3}{3!}+\&\text{c.}\right\}e^{2\alpha}$$

$$+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\alpha^n\left\{1-\frac{n+1}{n+1}(2\alpha)+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{(2\alpha)^2}{2!}-\frac{(n+1)(n+3)(n+5)}{(n+1)(n+2)(n+3)}\frac{(2\alpha)^3}{3!}+\&\text{c.}\right\}e^{2\alpha}$$

$$=\quad \tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\left\{1+\frac{n-1}{n-1}(2\alpha)+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{(2\alpha)^2}{2!}+\frac{(n-1)(n-3)(n-5)}{(n-1)(n-2)(n-3)}\frac{(2\alpha)^3}{3!}+\&\text{c.}\right\}e^{-2\alpha}$$

$$+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\alpha^n\left\{1+\frac{n+1}{n+1}(2\alpha)+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{(2\alpha)^2}{2!}+\frac{(n+1)(n+3)(n+5)}{(n+1)(n+2)(n+3)}\frac{(2\alpha)^3}{3!}+\&\text{c.}\right\}e^{-2\alpha},$$

the series extending to infinity in every case.

The method by which the fundamental formula (7) has been obtained is open to some objections. These will be noticed, and a complete proof of (7) given, in art. 28.

21. We have

$$\frac{\Gamma(-\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)}=-\frac{2}{n}\frac{\Gamma(1-\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)}=-\frac{2}{n}\frac{\pi}{\sin\frac{1}{2}n\pi}\frac{1}{\{\Gamma(\frac{1}{2}n)\}^2}, \text{ since } \Gamma(m)\Gamma(1-m)=\frac{\pi}{\sin m\pi};$$

and, if n is an uneven integer, this

$$=(-)^{\frac{1}{2}(n+1)}\frac{2\pi}{n}\frac{1}{\{\sqrt{\pi}.\frac{1}{2}.\frac{3}{2}.\frac{5}{2}\ldots\frac{1}{2}(n-2)\}^2}=(-)^{\frac{1}{2}(n+1)}\frac{2^n n}{1^2.3^2.5^2\ldots n^2}=g_1,$$

g_1 being as defined in art. 16, when a is put $=2$.

Now, if n is a positive uneven integer, we have, by art. 16, $U_1+g_1V_1=R_1'$; and $M+g_1N$ is equal to $U_1+g_1V_1$ when α is written for x, and a put $=2$; so that, if n is a positive uneven integer,

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\left\{1+\frac{n-1}{n-1}(2\alpha)+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{(2\alpha)^2}{2!}\right.$$
$$\left.+\frac{(n-1)(n-3)(n-5)}{(n-1)(n-2)(n-3)}\frac{(2\alpha)^3}{3!}+\&c.\right\}e^{-2\alpha},$$

the series terminating at the term preceding the first term containing a zero factor in the numerator.

Transforming the integral by assuming $x=\frac{\alpha}{x'}$, we find that, if n is a negative uneven integer,

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx=\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\alpha^n\left\{1+\frac{n+1}{n+1}(2\alpha)+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{(2\alpha)^2}{2!}+\&c.\right\}e^{-2\alpha},$$

the series terminating when the first zero factor appears.

Thus, generally,

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx=$$

$$\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\left\{1+\frac{n-1}{n-1}(2\alpha)+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{(2\alpha^2)}{2!}+\frac{(n-1)(n-3)(n-5)}{(n-1)(n-2)(n-3)}\frac{(2\alpha)^3}{3!}+\&c.\right\}e^{-2\alpha}$$

$$+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\alpha^n\left\{1+\frac{n+1}{n+1}(2\alpha)+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{(2\alpha)^2}{2!}+\frac{(n+1)(n+3)(n+5)}{(n+1)(n+2)(n+3)}\frac{(2\alpha)^3}{3!}+\&c.\right\}e^{-2\alpha}. \quad (8),$$

if n is not equal to an integer: but if n = a positive uneven integer, the first series continued up to the first term containing a zero factor in the numerator is the value of the integral, the second series being ignored altogether; and if n = a negative uneven integer, the second series continued up to the first term containing a zero factor in the numerator, is the value of the integral, the first series being ignored altogether. The rule may therefore be stated as follows: if neither series terminates then (8) represents the value of the integral, but if one of the series terminates, the finite series represents the value of the integral, the other being ignored; a series being supposed to terminate at the term preceding the first term that contains a zero factor in the numerator.

The apparent change of form is curious, but the reason for it has fully appeared in § I., arts. 3–6. In the 'British Association Report' for 1872 (Transactions of the

Sections, pp. 15–17) I gave the formula (8) with a brief indication of the method by which it had been obtained; this method is substantially the same as that just explained. As far as I know the general value of the integral had not been given before; although the value in the case of $n=$ an uneven integer has been long known. It is scarcely necessary to remark that in (8) n must not $=$ an even integer: this case is specially excepted throughout (see end of art. 1).

22. The case when $n=$ an uneven integer is included in a general formula given by CAUCHY in vol. i. of his 'Exercices des Mathématiques' (1826), pp. 54–56. He has there shown that if

$$P_{2i}=\int_0^\infty x^{2i}\phi(x)dx, \qquad Q_{2i}=\int_0^\infty x^{2i}\phi\left(x-\frac{1}{x}\right)dx,$$

i being a positive integer, and ϕ an even function, then

$$Q_{2i}=P_0+\frac{i(i+1)}{2!}P_2+\frac{(i-1)i(i+1)(i+2)}{4!}P_4 \ldots +\frac{2i-1}{1}P_{2i-2}+P_{2i} \quad . \quad . \quad (9).$$

This is the case corresponding to $a=1$ of a formula proved by BOOLE (Philosophical Transactions, vol. 147, 1857, p. 783), viz.

$$\int_0^\infty x^{2i}\phi\left(x-\frac{a}{x}\right)dx=\Sigma_{m=0}^{m=i}\frac{(2m+1)(2m+2)\ldots(i+m)}{(i-m)!}a^{i-m}\int_0^\infty x^{2m}\phi(x)dx \quad . \quad . \quad (10).$$

BOOLE's formula may however be deduced from CAUCHY's; for, replacing $\phi(x)$ by $\phi(ax)$, we have

$$Q_{2i}=\int_0^\infty x^{2i}\phi\left(ax-\frac{a}{x}\right)dx,$$

and this integral, transformed by assuming $x=\frac{x'}{b}$, becomes

$$\frac{1}{b^{2i}}\int_0^\infty x^{2i}\phi\left(\frac{ax}{b}-\frac{ab}{x}\right)dx,$$

in which, if we put $a=b$ and replace a^2 by a, the expression subject to the functional sign becomes $x-\frac{a}{x}$*

* In the 'Messenger of Mathematics,' vol. ii., p. 79, I stated that CAUCHY's proof was not applicable to the more general theorem in which $x-\frac{1}{x}$ was replaced by $x-\frac{a}{x}$. The error was corrected in a paper "On a Formula of CAUCHY's for the Evaluation of a Class of Definite Integrals" ('Proceedings of the Cambridge Philosophical Society,' vol. iii., pp. 5–12, 1876); this paper contains also the theorem, corresponding to CAUCHY's, in which ϕ is an uneven function.

To deduce the value of the integral $\int_0^\infty x^{2i}e^{-x^2-\frac{a^2}{x^2}}dx$ from BOOLE's formula, let $\phi(x)=e^{-x^2}$, then

$$P_{2i}=\int_0^\infty x^{2i}e^{-x^2}dx=\tfrac{1}{2}\Gamma(i+\tfrac{1}{2})=\tfrac{1}{2}.\sqrt{\pi}.\tfrac{1}{2}.\tfrac{3}{2}.\tfrac{5}{2}\ldots\frac{2i-1}{2};$$

and therefore the coefficient of a^{i-m}

$$=\frac{\sqrt{\pi}}{2}\,\frac{(2m+1)(2m+2)\ldots(i+m)}{(i+m)!}\,\tfrac{1}{2}.\tfrac{3}{2}\ldots\frac{2m-1}{2}=\frac{\sqrt{\pi}}{2}\,\frac{(i+m)!}{(i-m)!m!}\,\frac{1}{2^{2m}}.$$

Thus

$$\int_0^\infty x^{2i}e^{-\left(x-\frac{a}{x}\right)^2}dx=\frac{\sqrt{\pi}}{2}\left\{a^i+\frac{(i+1)!}{(i-1)!1!}\,\frac{a^{i-1}}{2^2}+\frac{(i+2)!}{(i-2)!2!}\,\frac{a^{i-2}}{2^4}\ldots+\frac{(2i)!}{i!}\,\frac{1}{2^{2i}}\right\}\quad . \quad (11),$$

whence

$$\int_0^\infty x^{2i}e^{-x^2-\frac{a^2}{x^2}}dx=\frac{\sqrt{\pi}}{2}a^i\left\{1+\frac{i(i+1)}{2}\left(\frac{1}{2a}\right)+\frac{(i-1)i(i+1)(i+2)}{2.4}\left(\frac{1}{2a}\right)^2+\&c.\right\}e^{-2a}\quad . \quad (12),$$

the series being continued till it terminates of itself.

This formula is in effect that given by CAUCHY ('Exercices,' *loc. cit.*, p. 55) for the evaluation of the integral. It had however, as CAUCHY himself remarks, been previously published by LEGENDRE in vol. i., p. 366, of his 'Exercices de Calcul Intégral' (1811). LEGENDRE, whose method is quite different to CAUCHY's, adds that EULER, in vol. iv., p. 415,* of his 'Institutiones Calculi Integralis' (1794) mentions the integrals

$$\int_0^\infty x^{-\frac{3}{2}}e^{-\frac{1+x^2}{2nx}}dx,\qquad \int_0^\infty x^{-\frac{5}{2}}e^{-\frac{1+x^2}{2nx}}dx,$$

which correspond to the particular cases $i=-1$ and $i=-2$ of the integral in (12), as apparently not admitting of evaluation by known methods; and he gives their values.

If in the series in (11) the terms be written in the reverse order, we have

$$\int_0^\infty x^{2i}e^{-x^2-\frac{a^2}{x^2}}dx=\frac{\sqrt{\pi}}{2}\,\frac{(2i)!}{2^{2i}i!}\left\{1+\frac{i}{2i}2^2a+\frac{i(i-1)}{2i(2i-1)}\,\frac{2^4a^2}{2!}\ldots+\frac{i!}{(2i)!}2^{2i}a^i\right\}e^{-2a},$$

which agrees with (8) when $n=2i+1$, since

$$\tfrac{1}{2}\Gamma(i+\tfrac{1}{2})=\frac{\sqrt{\pi}}{2}\,\frac{1.3\ldots(2i-1)}{2^i}=\frac{\sqrt{\pi}}{2}\,\frac{(2i)!}{2^{2i}i!}.$$

* 'Supplementum V.' ad tom. 1, cap. viii.

Transforming the integral in (12) by assuming $x=\frac{a}{x'}$, it is seen that (12) is true also when i = a negative integral; so that this formula is true when $i=$ any integer.

The same transformation shows that in general, if ϕ is an even function,

$$\int_0^\infty x^{-2i-2}\phi\left(x-\frac{a}{x}\right)dx=a^{-2i-1}\int_0^\infty x^{2i}\phi\left(x-\frac{a}{x}\right)dx.$$

Putting $2i=n-1$, the formula (12) assumes the form

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{a^2}{x^2}}dx=\frac{\sqrt{\pi}}{2}a^{\frac{1}{2}(n-1)}\left\{1+(n^2-1^2)\left(\frac{1}{16a}\right)+\frac{(n^2-1^2)(n^2-3^2)}{2!}\left(\frac{1}{16a}\right)^2+\&c.\right\}e^{-2a},$$

which is true when $n=$ an uneven integer, the series being continued till it terminates of itself.

23. The investigations of the formulæ (9) and (10) given by CAUCHY and BOOLE are only applicable in the case of i an integer, and do not indicate what the formulæ become when i is unrestricted. A method, however, which I have employed in the 'Messenger of Mathematics' (vol. ii., 1872, pp. 78, 79) to prove BOOLE's formula, and which depends on direct transformations of the integrals, leads to the general theorem.

We have

$$\int_0^\infty x^n\phi\left(x-\frac{a}{x}\right)dx=\int_{\sqrt{a}}^\infty x^n\phi\left(x-\frac{a}{x}\right)dx+\int_0^{\sqrt{a}} x^n\phi\left(x-\frac{a}{x}\right)dx\,;$$

and if we transform the second integral by assuming $x=\frac{\sqrt{a}}{x'}$, then, since ϕ is an even function, we find that the original integral

$$=\int_{\sqrt{a}}^\infty \phi\left(x-\frac{a}{x}\right)\left\{x^n+\frac{a^{n+1}}{x^{n+2}}\right\}dx$$

$$=\frac{1}{n+1}\int_{\sqrt{a}}^\infty \phi\left(x-\frac{a}{x}\right)\frac{d}{dx}\left[x^{n+1}-\frac{a^{n+1}}{x^{n+1}}\right]dx.$$

Now transform this integral by assuming $x-\frac{a}{x}=v$; we thus have $x=\frac{1}{2}\{v\pm\sqrt{(v^2+4a)}\}$, and, taking the upper sign, the integral becomes

$$\frac{1}{n+1}\int_0^\infty \phi(v)\frac{d}{dv}\left[\left\{\frac{v+\sqrt{(v^2+4a)}}{2}\right\}^{n+1}-\left\{\frac{-v+\sqrt{(v^2+4a)}}{2}\right\}^{n+1}\right]dv.$$

If n is an even integer, the quantity in square brackets

$$=\left\{\frac{v+\sqrt{(v^2+4a)}}{2}\right\}^{n+1}+\left\{\frac{v-\sqrt{(v^2+4a)}}{2}\right\}^{n+1}$$

$$=v^{n+1}\left\{1+(n+1)\frac{a}{v^2}+(n+1)\frac{n-2}{2!}\frac{a^2}{v^4}+(n+1)\frac{(n-3)(n-4)}{3!}\frac{a^3}{v^6}\ldots+(n+1)\frac{a^{\frac{1}{2}n}}{v^n}\right\},$$

this expression containing $\frac{1}{2}n+1$ terms.

Thus, if n is an even integer, we have

$$\int_0^\infty x^n\phi\left(x-\frac{a}{x}\right)dx=\int_0^\infty \phi(v)\left\{v^n+(n-1)av^{n-2}+\frac{(n-2)(n-3)}{2!}a^2v^{n-4}\ldots+a^{\frac{1}{2}n}\right\}dv,$$

which agrees with BOOLE's formula. But if n is not an even integer, the expression in square brackets when expanded contains an infinite number of terms, and putting, as before,

$$\mathrm{P}_n=\int_0^\infty x^n\phi(x)dx,\qquad \mathrm{Q}_n=\int_0^\infty x^n\phi\left(x-\frac{a}{x}\right)dx,$$

the general formula is

$$\mathrm{Q}_n=\mathrm{P}_n+(n-1)a\mathrm{P}_{n-2}+\frac{(n-2)(n-3)}{2!}a^2\mathrm{P}_{n-4}+\frac{(n-3)(n-4)(n-5)}{3!}a^3\mathrm{P}_{n-6}+\text{\&c. } ad\ inf.$$

$$+a^{n+1}\mathrm{P}_{-n-2}\quad+(n+3)a^{n+2}\mathrm{P}_{-n-4}\quad+\frac{(n+4)(n+5)}{2!}a^{n+3}\mathrm{P}_{-n-6}\quad+\text{\&c. } ad\ inf.$$

24. This formula involves infinite terms unless ϕ is such a function that the integrals P_{n-2}, P_{n-4}, ... P_{-n-2}, P_{-n-6} ... are all finite. This condition is not fulfilled when $\phi(x)=e^{-x^2}$, for $\int_0^\infty x^ne^{-x^2}dx$ is infinite when $n=$ or <-1, so that we do not obtain by means of the formula a demonstration of the equation (8). If however we replace $\int_0^\infty x^ne^{-x^2}dx$ by $\Gamma\left(\frac{n+1}{2}\right)$ in all the terms, whether the integral be really infinite or not, we do in fact, as we should expect, obtain (8). For, putting $\phi(x)=e^{-x^2}$, substituting gamma-functions for the integrals, and writing $n-1$ in place of n, the formula gives

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{a^2}{x^2}}dx=\tfrac{1}{2}\left\{\Gamma(\tfrac{1}{2}n)\quad+(n-2)a\Gamma(\tfrac{1}{2}n-1)\quad+\frac{(n-3)(n-4)}{2!}a^2\Gamma(\tfrac{1}{2}n-2)\quad+\text{\&c.}\right\}e^{-2a}$$

$$+\tfrac{1}{2}a^n\left\{\Gamma(-\tfrac{1}{2}n)-(n+2)a\Gamma(-\tfrac{1}{2}n-1)+\frac{(n+3)(n+4)}{2!}a^2\Gamma(-\tfrac{1}{2}n-2)+\text{\&c.}\right\}e^{-2a}$$

$$=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\left\{1+\frac{n-2}{n-2}(2a)+\frac{(n-3)(n-4)}{(n-2)(n-4)}\frac{(2a)^2}{2!}+\frac{(n-4)(n-5)(n-6)}{(n-2)(n-4)(n-6)}\frac{(2a)^3}{3!}\quad+\text{\&c.}\right\}e^{-2a}$$

$$+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)a^n\left\{1+\frac{n+2}{n+2}(2a)+\frac{(n+3)(n+4)}{(n+2)(n+4)}\frac{(2a)^2}{2!}+\frac{(n+4)(n+5)(n+6)}{(n+2)(n+4)(n+6)}\frac{(2a)^3}{3!}\quad+\text{\&c.}\right\}e^{-2a}.$$

The coefficients are readily identified with those in (8), for evidently

$$\frac{(n+r+1)(n+r+2)\ldots(n+2r)}{(n+2)(n+4)\ldots(n+2r)}=\frac{(n+1)(n+3)\ldots(n+2r-1)}{(n+1)(n+2)\ldots(n+r)}.$$

This process, regarded as a method of obtaining the formula (8), is of course unsound, and could not be rendered satisfactory without careful discussion and development. Such substitutions, however, very frequently give correct results, and it is generally interesting to examine whether, in any case that arises, the result so derived is true or not. In this instance also we thus obtain two new forms of the expression forming the right hand member of (8).

25. Transforming (8) by the assumption $x=v^{\frac{1}{n}}$, putting $m=\frac{2}{n}$, and writing a for α, it will be found that

$$\int_0^\infty e^{-v^m-\frac{a^2}{v^m}}dv=\Gamma\left(1+\frac{1}{m}\right)\left\{1+\frac{m-2}{m-2}(2a)+\frac{(m-2)(3m-2)}{(m-2)(2m-2)}\frac{(2a)^2}{2!}+\&c.\right\}e^{-2a}$$

$$-\Gamma\left(1-\frac{1}{m}\right)a^{\frac{2}{m}}\left\{1+\frac{m+2}{m+2}(2a)+\frac{(m+2)(3m+2)}{(m+2)(2m+2)}\frac{(2a)^2}{2!}+\&c.\right\}e^{-2a}. \quad . \ (13);$$

where, as before, if either series terminates through the presence of a zero factor in a numerator, the terminating series represents the value of the integral, and if neither series terminates, both are to be included. When one of the series terminates, that is, when m = twice the reciprocal of an uneven integer, the formula may, by taking the terms of the series in the reverse order, be written

$$\int_0^\infty e^{-v^m-\frac{a^2}{v^m}}dv=\frac{\sqrt{\pi}}{m}a^{\frac{1}{m}-\frac{1}{2}}\left\{1-\frac{m^2-2^2}{m^2}\left(\frac{1}{16a}\right)+\frac{(m^2-2^2)(3^2m^2-2^2)}{m^2.2m^2}\left(\frac{1}{16a}\right)^2-\&c.\right\}e^{-2a}.$$

Putting $v=\alpha^{\frac{2}{m}}x$ and $a^2=\alpha^2\beta^2$, (13) becomes

$$\int_0^\infty e^{-\alpha^2x^m-\frac{\beta^2}{x^m}}dx=\Gamma\left(1+\frac{1}{m}\right)\alpha^{-\frac{2}{m}}\left\{1+\frac{m-2}{m-2}(2\alpha\beta)+\frac{(m-2)(3m-2)}{(m-2)(2m-2)}\frac{(2\alpha\beta)^2}{2!}+\&c.\right\}e^{-2\alpha\beta}$$

$$-\Gamma\left(1-\frac{1}{m}\right)\beta^{\frac{2}{m}}\left\{1+\frac{m+2}{m+2}(2\alpha\beta)+\frac{(m+2)(3m+2)}{(m+2)(2m+2)}\frac{(2\alpha\beta)^2}{2!}+\&c.\right\}e^{-2\alpha\beta}.$$

If for example $m=2$, we have the well-known result

$$\int_0^\infty e^{-\alpha^2x^2-\frac{\beta^2}{x^2}}dx=\Gamma(\tfrac{3}{2})\alpha^{-1}e^{-2\alpha\beta}=\frac{\sqrt{\pi}}{2\alpha}e^{-2\alpha\beta}.$$

26. The definite integral

$$\int_0^\infty \frac{\cos bx}{(a^2+x^2)^n}dx$$

has been evaluated when n is a positive integer,* the formula in this case being

$$\int_0^\infty \frac{\cos bx}{(a^2+x^2)^n}dx=\frac{\pi}{2^n(n-1)!}\frac{b^{n-1}}{a^n}\left\{1+\frac{n(n-1)}{2}\left(\frac{1}{ab}\right)+\frac{(n+1)n(n-1)(n-2)}{2.4}\left(\frac{1}{ab}\right)^2+\&c.\right\}e^{-ab} \quad (14).$$

* See SCHLÖMILCH, 'Analytische Studien' (Leipzig, 1848), part ii., p. 97, and CRELLE'S Journal, vol. xxxiii., p. 273, or CATALAN, 'LIOUVILLE'S Journal,' ser. 1, vol. v., p. 110.

This result may be readily obtained by differentiating both members of the equation

$$\int_0^\infty \frac{\cos bx}{(a^2+x^2)^n}\,dx=\frac{\pi}{2a}\,e^{-ab}$$

$n-1$ times with regard to a^2: see *infra*, art. 31. I now proceed to investigate the value of the integral when n is unrestricted: it is to be observed, however, that n must be positive and greater than unity, for otherwise the integral is infinite in value.

It is easy to prove that the integral

$$u=x^p\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^p}\,d\xi$$

satisfies the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p-1)}{x^2}\,u\,;$$

for, by actual differentiation,

$$\frac{d^2u}{dx^2}-\frac{p(p-1)}{x^2}u=2px^p\int_0^\infty (x^2-\xi^2-2p\xi^2)\frac{\cos a\xi}{(x^2+\xi^2)^{p+2}}\,d\xi\,;$$

and by a double integration by parts we find that

$$x^p\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^p}\,d\xi=\frac{2p}{a^2}\,x^p\int_0^\infty (x^2-\xi^2-2p\xi^2)\,\frac{\cos a\xi}{(x^2+\xi^2)^{p+2}}\,d\xi.*$$

Thus the value of the integral $\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}\,d\xi$ must be of the form $x^{-p-1}(\mathrm{AU}+\mathrm{BV})$, where U and V are as defined in art. 3, and A and B are constants to be determined. It is however more convenient to avoid the determination of the constants by deducing the value of the integral from the formula (8) of art. 21.

In the 'Journal de l'École Polytechnique,' Cah. xvi. (vol. ix.), p. 241, POISSON has proved a formula which, after some unimportant transformations, may be written

$$\int_0^\infty x^{2n}e^{-x^2-\frac{b^2}{x^2}}\,dx=\frac{\Gamma(n+1)}{\sqrt{\pi}}\int_0^\infty \frac{\cos 2bx}{(1+x^2)^{n+1}}\,dx\,;$$

POISSON's demonstration holds good for all values of n such that the integral upon the right-hand side of the equation is finite. Putting $n-1$ for n and transforming the right-hand integral by assuming $x=\frac{x'}{a}$, this equation becomes

* 'Quarterly Journal of Mathematics,' vol. xii., p. 130.

$$\int_0^\infty x^{2n-2}e^{-x^2-\frac{b^2}{x^2}}\,dx=a^{2n-1}\frac{\Gamma(n)}{\sqrt{\pi}}\int_0^\infty\frac{\cos\left(\frac{2bx}{a}\right)}{(a^2+x^2)^n}\,dx\,;$$

whence, replacing $\frac{2b}{a}$ by b,

$$\int_0^\infty\frac{\cos bx}{(a^2+x^2)^n}\,dx=\frac{\sqrt{\pi}}{\Gamma(n)}a^{-2n+1}\int_0^\infty x^{2n-2}\;e^{-x^2-\frac{a^2b^2}{4x^2}}\,dx\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\qquad(15)$$

$$=\frac{\sqrt{\pi}}{\Gamma(n)}a^{-2n+1}\Big[\tfrac{1}{2}\Gamma(n-\tfrac{1}{2})\Big\{1+\frac{2n-2}{2n-2}ab+\frac{(2n-2)(2n-4)}{(2n-2)(2n-3)}\frac{(ab)^2}{2!}$$
$$+\frac{(2n-2)(2n-4)(2n-6)}{(2n-2)(2n-3)(2n-4)}\frac{(ab)^3}{3!}+\&\text{c.}\Big\}$$
$$+\tfrac{1}{2}\Gamma(-n+\tfrac{1}{2})\left(\frac{ab}{2}\right)^{2n-1}\Big\{1+\frac{2n}{2n}ab+\frac{2n(2n+2)}{2n(2n+1)}\frac{(ab)^2}{2!}$$
$$+\frac{2n(2n+2)(2n+4)}{2n(2n+1)(2n+2)}\frac{(ab)^3}{3!}+\&\text{c.}\Big\}\Big]e^{-ab}$$
$$=\tfrac{1}{2}\frac{\sqrt{\pi}}{\Gamma(n)}\Big[\Gamma(n-\tfrac{1}{2})a^{-2n+1}\Big\{1+\frac{2n-2}{2n-2}ab+\frac{(2n-2)(2n-4)}{(2n-2)(2n-3)}\frac{(ab)^2}{2!}+\&\text{c.}\Big\}$$
$$+\Gamma(-n+\tfrac{1}{2})(\tfrac{1}{2}b)^{2n-1}\Big\{1+\frac{2n}{2n}ab+\frac{2n(2n+2)}{2n(2n+1)}\frac{(ab)^2}{2!}+\&\text{c.}\Big\}\Big]e^{-ab},$$

which represents the value of the integral for all values of n greater than unity. If n is a positive integer the first series terminates through the presence of a zero factor in a numerator, and this finite series is the value of the integral, the second series being ignored.

If n is a positive integer, then, writing the terms of the series in the reverse order,

$$\int_0^\infty\frac{\cos bx}{(a^2+x^2)^n}\,dx=\tfrac{1}{2}\frac{\sqrt{\pi}}{\Gamma(n)}\Gamma(n-\tfrac{1}{2})a^{-2n+1}\Big\{\frac{(n-1)!}{(2n-2)\ldots n}\frac{(2ab)^{n-1}}{(n-1)!}$$
$$+\frac{(n-1)\ldots 2}{(2n-2)\ldots(n+1)}\frac{(2ab)^{n-2}}{(n-2)!}\ldots+\frac{n-1}{2n-2}(2ab)+1\Big\}e^{-ab}$$
$$=\frac{\sqrt{\pi}}{\Gamma(n)}\frac{b^{n-1}}{a^n}\frac{(n-\frac{3}{2})(n-\frac{5}{2})\ldots\frac{1}{2}\cdot\sqrt{\pi}}{(2n-2)\ldots n}2^{n-2}\Big\{1+n(n-1)\left(\frac{1}{2ab}\right)$$
$$+\frac{(n+1)n(n-1)(n-2)}{2!}\left(\frac{1}{2ab}\right)^2\ldots+\frac{(2n-2)!}{(n-1)!}\left(\frac{1}{2ab}\right)^{n-1}\Big\}e^{-ab}$$
$$=\frac{\pi}{\Gamma(n)}\frac{1}{2^n}\frac{b^{n-1}}{a^n}\Big\{1+\frac{n(n-1)}{2}\left(\frac{1}{ab}\right)+\frac{(n+1)n(n-1)(n-}{2.4}\left(\frac{1}{ab}\right)^2+\&\text{c.}\Big\}e^{\;ab},$$

which agrees with (14).

27. In a similar manner we may obtain the value of the integral

$$\int_0^\infty \frac{x \sin bx}{(a^2+x^2)^n} dx$$

which is finite for all values of n greater than unity. For, differentiating (15) with respect to b, we have

$$\int_0^\infty \frac{x \sin bx}{(a^2+x^2)^n} dx = \tfrac{1}{2}\frac{\sqrt{\pi}}{\Gamma(n)} a^{-2n+3} b \int_0^\infty x^{2n-4} e^{-x^2-\frac{a^2b^2}{4x^2}} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$= \tfrac{1}{4}\frac{\sqrt{\pi}}{\Gamma(n)} b \Big[\Gamma(n-\tfrac{3}{2}) a^{-2n+3} \Big\{ 1 + \frac{2n-4}{2n-4} ab + \frac{(2n-4)(2n-6)}{(2n-4)(2n-5)} \frac{(ab)^2}{2!} + \&c. \Big\}$$

$$+ \Gamma(-n+\tfrac{3}{2})(\tfrac{1}{2}b)^{2n-3} \Big\{ 1 + \frac{2n-2}{2n-2} ab + \frac{(2n-2)(2n)}{(2n-2)(2n-1)} \frac{(ab)^2}{2!} + \&c. \Big\} \Big] e^{-ab},$$

where, as before, if the first series terminates, the finite portion of it represents the value of the integral.

If n is a positive integer, and the terms of the series are written in the reverse order, we find

$$\int_0^\infty \frac{x \sin bx}{(a^2+x^2)^n} dx = \frac{\pi}{2^n(n-1)!} \frac{b^{n-1}}{a^{n-1}} \Big\{ 1 + \frac{(n-1)(n-2)}{2}\Big(\frac{1}{ab}\Big) + \frac{n(n-1)(n-2)(n-3)}{2.4}\Big(\frac{1}{ab}\Big)^2 + \&c. \Big\} e^{-ab},$$

which is a known result (SCHLÖMILCH, 'Anal. Stud.,' *loc. cit.*, p. 97).

It follows at once by combining (15) and (16) that, for all values of n such that the integrals are not infinite, viz., if $n =$ or > 1,

$$\int_0^\infty \frac{x \sin bx}{(a^2+x^2)^{n+1}} dx = \tfrac{1}{2}\frac{b}{n} \int_0^\infty \frac{\cos bx}{(a^2+x^2)^n} dx.$$

It would be strange if this equation were new, but I have not met with it anywhere: it is readily proved in the case of n an integer, for

$$-2a \int_0^\infty \frac{x \sin bx}{(a^2+x^2)^2} dx = \frac{d}{da}(\tfrac{1}{2}\pi e^{-ab}) = -\tfrac{1}{2}\pi b e^{-ab},$$

and

$$\int_0^\infty \frac{\cos bx}{a^2+x^2} dx = \frac{\pi}{2a} e^{-ab}$$

whence

$$\int_0^\infty \frac{x \sin bx}{(a^2+x^2)^2} dx = \tfrac{1}{2} b \int_0^\infty \frac{\cos bx}{a^2+x^2} dx,$$

which, differentiated n times with regard to a^2, gives the relation in question.

28. The method by which the formula (7) was obtained in art. 20 is not satisfactory for two reasons, (i) because the integral $\int_0^\infty x^{n-1}e^{-x}dx$ is infinite in value when n is negative, while the gamma-function, which is supposed to satisfy the equation $\Gamma(n+1) = n\Gamma(n)$ for all values of n, is finite when n is negative, except when n is a negative integer, so that we are not entitled to assume that we may always replace the integral by the gamma-function, and (ii), because it is assumed that we may change the sign of n in the equation giving the value of the integral. The following demonstration of the formula (7) is, however, I believe, quite rigorous.

The gamma-function is supposed to be defined by the equation $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}dx$ from $n=0$ to $n=1$, and by the equation $\Gamma(n+1)=n\Gamma(n)$ for all other values of n. This is in effect the definition of the gamma-function generally adopted in analysis.

We have seen in art. 20 that

$$y=\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx$$

satisfies the differential equation $\frac{d^2y}{d\alpha^2}-\frac{n-1}{\alpha}\frac{dy}{dx}-4y=0$, so that $y=\mathrm{AM}+\mathrm{B}\alpha^n\mathrm{N}$; and by a simple transformation of the integral, it follows that

$$\int_0^\infty x^{n-1}e^{-\alpha x^2-\frac{\beta}{x^2}}dx=\mathrm{A}\alpha^{-\frac{1}{2}n}\mathrm{H}_n+\mathrm{B}\beta^{\frac{1}{2}n}\mathrm{K}_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

where

$$\mathrm{H}_n=1-\frac{1}{n-2}(2\alpha\beta)+\frac{1}{(n-2)(n-4)}\frac{(2\alpha\beta)^2}{2!}-\frac{1}{(n-2)(n-4)(n-6)}\frac{(2\alpha\beta)^3}{3!}+\&c.$$

$$\mathrm{K}_n=1+\frac{1}{n+2}(2\alpha\beta)+\frac{1}{(n+2)(n+4)}\frac{(2\alpha\beta)^2}{2!}+\frac{1}{(n+2)(n+4)(n+6)}\frac{(2\alpha\beta)^3}{3!}+\&c.$$

Suppose n to be intermediate to 0 and 1. Put $\beta=0$ in (17) and we have $\int_0^\infty x^{n-1}e^{-\alpha x^2}dx = \mathrm{A}\alpha^{-\frac{1}{2}n}$, whence $\mathrm{A}=\frac{1}{2}\Gamma(\frac{1}{2}n)$.

Now by actual differentiation of the series represented by H_n and K_n we find that

$$\frac{d}{d\alpha}(\alpha^{-\frac{1}{2}n}\mathrm{H}_n)=-\tfrac{1}{2}n\alpha^{-\frac{1}{2}n-1}\mathrm{H}_{n+2}, \qquad \frac{d\mathrm{K}_n}{d\alpha}=\frac{2\beta}{n+2}\mathrm{K}_{n+2},$$

and similarly

$$\frac{d\mathrm{H}_n}{d\beta}=-\frac{2\alpha}{n-2}\mathrm{H}_{n-2}, \qquad \frac{d}{d\beta}(\beta^{\frac{1}{2}n}\mathrm{K}_n)=\tfrac{1}{2}n\beta^{\frac{1}{2}n-1}\mathrm{K}_{n-2}.$$

Transforming the integral in (17) by assuming $x=\frac{1}{x'}$, it becomes

$$\int_0^\infty x^{-n-1}e^{-\beta x^2-\frac{\alpha}{x^2}}dx=\mathrm{A}\alpha^{-\frac{1}{2}n}\mathrm{H}_n+\mathrm{B}\beta^{\frac{1}{2}n}\mathrm{K}_n;$$

whence, differentiating with regard to β,

$$\int_0^\infty x^{-n+1}e^{-\beta x^2-\frac{\alpha}{x^2}}dx = -\mathrm{A}\alpha^{-\frac{1}{2}n}\frac{d\mathrm{H}_n}{d\beta} - \mathrm{B}\frac{d}{d\beta}(\beta^{\frac{1}{2}n}\mathrm{K}_n),$$

$$= \frac{2\alpha^{-\frac{1}{2}n+1}}{n-2}\mathrm{AH}_{n-2} - \tfrac{1}{2}n\beta^{\frac{1}{2}n-1}\mathrm{BK}_{n-2}.$$

Now $-n+1$ lies between 0 and 1, and putting $\alpha=0$, we have

$$\int_0^\infty x^{-n+1}e^{-\beta x^2} = \tfrac{1}{2}\beta^{\frac{1}{2}n-1}\Gamma(-\tfrac{1}{2}n+1) = -\tfrac{1}{2}n\beta^{\frac{1}{2}n-1}\mathrm{B},$$

giving

$$\mathrm{B} = -\frac{1}{n}\Gamma(-\tfrac{1}{2}n+1) = \tfrac{1}{2}\Gamma(-\tfrac{1}{2}n).$$

Thus, if n lies between 0 and 1, it has been proved that

$$\int_0^\infty x^{n-1}e^{-\alpha x^2-\frac{\beta}{x^2}}dx = \tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\alpha^{-\frac{1}{2}n}\mathrm{H}_n + \tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\beta^{\frac{1}{2}n}\mathrm{K}_n,$$

and this equation can be readily shown to be true for all values of n by differentiating both members of it any number of times with regard to α or β.

For, differentiating with regard to α,

$$\int_0^\infty x^{n+1}e^{-\alpha x^2-\frac{\beta}{x^2}}dx = \tfrac{1}{2}\Gamma(\tfrac{1}{2}n).\tfrac{1}{2}n\,\alpha^{-\frac{1}{2}n-1}\mathrm{H}_{n+2} - \tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\quad\frac{2\beta^{\frac{1}{2}n+1}}{n+2}\mathrm{K}_{n+2}$$

$$= \tfrac{1}{2}\Gamma(\tfrac{1}{2}n+1)\alpha^{-\frac{1}{2}n-1}\mathrm{H}_{n+2} + \tfrac{1}{2}\Gamma(-\tfrac{1}{2}n-1)\beta^{\frac{1}{2}n+1}\mathrm{K}_{n+2},$$

and, differentiating with regard to β,

$$\int_0^\infty x^{n-3}e^{-\alpha x^2-\frac{\beta}{x^2}}dx = \tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\quad\frac{2\alpha^{-\frac{1}{2}n+1}}{n-2}\mathrm{H}_{n-2} - \tfrac{1}{2}\Gamma(-\ \tfrac{1}{2}n)\tfrac{1}{2}n\beta^{\frac{1}{2}n-1}\mathrm{K}_{n-2}$$

$$= \tfrac{1}{2}\Gamma(\tfrac{1}{2}n-1)\alpha^{-\frac{1}{2}n+1}\mathrm{H}_{n-2} + \tfrac{1}{2}\Gamma(-\tfrac{1}{2}n+1)\beta^{\frac{1}{2}n-1}\mathrm{K}_{n-2}.$$

If, therefore, the formula (7) is true when $n=r$, it is true when $n=r\pm1$; and it has been proved to be true for all values of n between 0 and 1: it is therefore true for all real values of n.

It may be remarked that whatever value n may have, the integral is never infinite: so that the differentiations with regard to α or β are always permissible.

§ VI.

Symbolic forms of the particular integrals in the cases in which the differential equations admit of integration in a finite form. Arts. 29–42.

29. It has been shown in art. 26 that

$$u=x^{p+1}\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

satisfies the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p+1)}{x^2}u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Now

$$\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}\,d\xi=-\frac{1}{2}\frac{p}{x}\frac{d}{dx}\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^p}\,d\xi,$$

and therefore, if p is a positive integer,

$$\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}\,d\xi=(-)^p.p!\frac{1}{2^p}\left(\frac{1}{x}\frac{d}{dx}\right)^p\int_0^\infty \frac{\cos a\xi}{x^2+\xi^2}d\xi$$

$$=(-)^p.p!\frac{\pi}{2^{p+1}}\left(\frac{1}{x}\frac{d}{dx}\right)^p\frac{e^{-ax}}{x}.$$

The complete integral of (1) is therefore

$$u=x^{p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^p\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19);$$

and, since $\frac{1}{x}\frac{d}{dx}e^{ax}=\frac{ae^{ax}}{x}$, this result may be written also

$$u=x^{p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{p+1}(c_1e^{ax}+c_2e^{-ax}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Since the differential equation (1) remains unaltered if $-p-1$ is substituted for p, it follows that the complete integral of (1) may be expressed also in the forms

$$u=x^{-p}\left(\frac{1}{x}\frac{d}{dx}\right)^{-p-1}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right),$$

and

$$u=x^{-p}\left(\frac{1}{x}\frac{d}{dx}\right)^{-p}(c_1e^{ax}+c_2e^{-ax}).$$

30. Putting $u=x^{-p}v$, we see that the complete integral of the differential equation

$$\frac{d^2v}{dx^2}-\frac{2p}{x}\frac{dv}{dx}-a^2v=0,$$

when p is an integer, either positive or negative, is given by any one of the formulæ

$$v=x^{2p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{p+1}(c_1e^{ax}+c_2e^{-ax}),$$

$$v=\left(\frac{1}{x}\frac{d}{dx}\right)^{-p}(c_1e^{ax}+c_2e^{-ax}),$$

$$v=x^{2p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{p}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right),$$

$$v=\left(\frac{1}{x}\frac{d}{dx}\right)^{-p-1}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right).$$

Putting now $x=nz^{\frac{1}{n}}$, where $n=2p+1$ and $q=\frac{1}{n}$, the differential equation becomes

$$\frac{d^2v}{dz^2}-a^2z^{2q-2}v=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4);$$

and the integrals take the forms

$$v=z\left(z^{-2q+1}\frac{d}{dz}\right)^{p+1}\left(c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}\right),$$

$$v=\left(z^{-2q+1}\frac{d}{dz}\right)^{-p}\left(c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}\right),$$

$$v=z\left(z^{-2q+1}\frac{d}{dz}\right)^{p}\left(\frac{c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}}{z^q}\right),$$

$$v=\left(z^{-2q+1}\frac{d}{dz}\right)^{-p-1}\left(\frac{c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}}{z^q}\right).$$

If p is a positive integer $=i$, so that $q=\frac{1}{2i+1}$, then, from the first and third forms,

$$v=z\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1}\left(c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}\right),$$

or

$$v=z\left(z^{-2q+1}\frac{d}{dz}\right)^{i}\left(\frac{c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}}{z^q}\right);$$

and, if p is a negative integer $=-i-1$, so that $q=-\frac{1}{2i+1}$, then, from the second and fourth forms,

$$v=\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1}\left(c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}\right),$$

or

$$v=\left(z^{-2q+1}\frac{d}{dz}\right)^{i}\left(\frac{c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}}{z^q}\right).$$

31. These formulæ may be readily connected with the series-integrals found in § I., for, comparing (20) with the series P′ in arts. 3 and 5, we see that

$$x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{-ax}=\mathrm{A}x^{-i}\left\{1-\frac{i}{i}ax+\frac{i(i-1)}{i(i-\frac{1}{2})}\frac{a^2x^2}{2!}-\frac{i(i-1)(i-2)}{i(i-\frac{1}{2})(i-1)}\frac{a^3x^3}{3!}+\&\text{c.}\right\}e^{ax},$$

where A is a constant. Putting $i-1$ in place of i, this equation becomes

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i}e^{ax}=\mathrm{A}x^{-2i+1}\left\{1-\frac{i-1}{i-1}ax+\frac{(i-1)(i-2)}{(i-1)(i-\frac{3}{2})}\frac{a^2x^2}{2!}-\frac{(i-1)(i-2)(i-3)}{(i-1)(i-\frac{3}{2})(i-2)}\frac{a^3x^3}{3!}+\&\text{c.}\right\}e^{ax};$$

and, observing that the coefficient of $x^{-i}e^{ax}$ is a^i, it is evident that

$$\mathrm{A}=(-)^{i-1}\frac{i(i+1)\ldots(2i-2)}{2^{i-1}}a.$$

Writing the terms in the reverse order, as in § IV., we find

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i}e^{ax}=\left(\frac{a}{x}\right)^{i}\left\{1-\frac{i(i-1)}{2}\frac{1}{ax}+\frac{(i+1)i(i-1)(i-2)}{2.4}\frac{1}{a^2x^2}-\&\text{c.}\right\}e^{ax}\quad . \quad . \quad (21);$$

that is, on replacing x by $\sqrt{x}$,

$$\left(\frac{d}{dx}\right)^{i}e^{a\sqrt{x}}=\left(\frac{a}{2\sqrt{x}}\right)^{i}\left\{1-\frac{i(i-1)}{2}\frac{1}{a\sqrt{x}}+\frac{(i+1)i(i-1)(i-2)}{2.4}\frac{1}{a^2x}-\&\text{c.}\right\}e^{a\sqrt{x}},$$

which is a known formula (see, for example, SCHLÖMILCH's 'Analytische Studien' (1848), p. 86).

The formulæ which result from comparing the solutions of RICCATI's equation (4) in arts. 17 and 30 are

$$z\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1}e^{\frac{a}{q}z^q}=$$

$$(-)^i\frac{q(q-1)2q(2q-1)\ldots iq(iq-1)}{(q-1)(3q-1)\ldots\{(2i-1)q-1\}}a\left\{1-\frac{q-1}{q(q-1)}az^q+\frac{(q-1)(3q-1)}{q(q-1)2q(2q-1)}a^2z^{2q}-\&\text{c.}\right\}e^{\frac{a}{q}z^q}\quad (22),$$

if $q=\frac{1}{2i+1}$, and

$$\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1}e^{\frac{a}{q}z^q}=$$

$$(-)^i\frac{q(q+1)2q(2q+1)\ldots iq(iq+1)}{(q+1)(3q+1)\ldots\{(2i-1)q+1\}}az\left\{1-\frac{q+1}{(q+1)q}az^q+\frac{(q+1)(3q+1)}{(q+1)q(2q+1)2q}a^2z^{2q}-\&c.\right\}e^{\frac{a}{q}z^q}\ (23),$$

if $q=-\frac{1}{2i+1}$.

In the case of $q=\frac{1}{2i+1}$, we have $(2i+1)q-1=0$, and therefore

$$(2i-1)q-1=-2q,\qquad (2i-3)q-1=-4q,\quad\ldots\quad q-1=-2iq\,;$$

also

$$iq=\tfrac{1}{2}-\tfrac{1}{2}q,\qquad iq-1=-\tfrac{1}{2}-\tfrac{1}{2}q,\ \text{so that}\ iq(iq-1)=\tfrac{1}{4}(q^2-1),$$

and similarly

$$(i-1)q\{(i-1)q-1\}=\tfrac{1}{4}(3^2q^2-1),\quad \&c.$$

Thus the q-coefficient which multiplies the right-hand side of (22)

$$=\frac{(q^2-1)(3^2q^2-1)(5^2q^2-1)\ldots\{(2i-1)^2q^2-1\}}{(-)^i8^iq.2q.3q\ldots iq};$$

and, writing the terms on the right-hand side of (22) in the reverse order, the formula becomes

$$z\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1}e^{\frac{a}{q}z}=a^{i+1}z^{iq}\quad\left\{1+\frac{q^2-1}{q}\left(\frac{1}{8az^q}\right)+\frac{(q^2-1)(3^2q^2-1)}{q.2q}\left(\frac{1}{8az^q}\right)^2+\&c.\right\}e^{\frac{a}{q}z^q},$$

viz.

$$z\left(z^{-2q+1}\frac{d}{dz}\right)^{\frac{q+1}{2q}}e^{\frac{a}{q}z^q}=a^{\frac{q+1}{2q}}z^{\frac{1}{2}(1-q)}\left\{1+\frac{q^2-1}{q}\left(\frac{1}{8az^q}\right)+\frac{(q^2-1)(3^2q^2-1)}{q.2q}\left(\frac{1}{8az^q}\right)^2+\&c.\right\}e^{\frac{a}{q}z^q},$$

where $q=\frac{1}{2i+1}$.

Treating the formula (23) in the same manner, we find

$$\left(z^{-2q+1}\frac{d}{dz}\right)^{\frac{q-1}{2q}}e^{\frac{a}{q}z^q}=a^{\frac{q-1}{2q}}z^{\frac{1}{2}(1-q)}\left\{1+\frac{q^2-1}{q}\left(\frac{1}{8az^q}\right)+\frac{(q^2-1)(3^2q^2-1)}{q.2q}\left(\frac{1}{8az^q}\right)^2+\&c.\right\}e^{\frac{a}{q}z^q},$$

where $q=-\frac{1}{2i+1}$.

The right-hand members of these two formulæ differ from one another and from the last expression in § IV. (art. 19) only by the powers of a which occur as factors in the two former expressions.

32. It follows from the forms of u in art. 29 that

$$x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=\mathrm{A}x^{-i}\left(\frac{1}{x}\frac{d}{dx}\right)^{-i}e^{ax},$$

and it can be readily verified that $\mathrm{A}=a^{2i+1}$, so that we have

$$x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=a^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{-i}e^{ax} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

Transforming this result by putting $x=\frac{1}{q}z^q$ (q unrestricted), it becomes

$$z^{(2i+1)q}\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1}e^{\frac{a}{q}z^q}=a^{2i+1}\left(z^{-2q+1}\frac{d}{dz}\right)^{-i}e^{\frac{a}{q}z^q}.$$

If now $q=\frac{1}{2i+1}$, this may be written

$$z\left(z^{-2q+1}\frac{d}{dz}\right)^{\frac{q+1}{2q}}e^{\frac{a}{q}z^q}=a^{\frac{1}{q}}\left(z^{-2q+1}\frac{d}{dz}\right)^{\frac{q-1}{2q}}e^{\frac{a}{q}z^q},$$

and, on putting $q=-\frac{1}{2i+1}$, we obtain the same result; so that this formula holds good whenever q is of the forms $\pm\frac{1}{2i+1}$.

It follows from this theorem and from the two formulæ at the end of the last article that

$$z\left(z^{-2q+1}\frac{d}{dz}\right)^{\frac{q+1}{2q}}e^{\frac{a}{q}z^q}=a^{\frac{1}{q}}\left(z^{-2q+1}\frac{d}{dz}\right)^{\frac{q-1}{2q}}e^{\frac{a}{q}z^q}$$

$$=a^{\frac{q+1}{2q}}z^{\frac{1}{2}(1-q)}\left\{1-\frac{1-q^2}{q}\cdot\frac{1}{8az^q}+\frac{(1-q^2)(1-3^2q^2)}{q.2q}\left(\frac{1}{8az^q}\right)^2-\&c.\right\}e^{\frac{a}{q}z^q},$$

where $q=\pm\frac{1}{2i+1}$.

The relation (24), or, as it may be written more conveniently,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i}x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=a^{2i+1}e^{ax} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25),$$

admits of being established as follows.

Suppose e^{ax} expanded in ascending powers of x, and consider the term in x^p: we have

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}x^p=p(p-2)\ldots(p-2i)x^{p-2i-2},$$

and

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{2i+1}.x^{p-2i-2}=\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{p-1}=(p-1)(p-3)\ldots(p-2i+1)x^{p-2i},$$

so that

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1} x^p=p(p-1)(p-2)\ldots(p-2i)x^{p-2i}$$

$$=\left(\frac{d}{dx}\right)^{2i+1} x^p\,;$$

and therefore,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1} e^{ax}=\left(\frac{d}{dx}\right)^{2i+1} e^{ax},$$

$$=a^{2i+1}e^{ax}.$$

The preceding investigation shows also that, if $\phi(x)$ denotes any function of x, then

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1} \phi(x)=\left(\frac{d}{dx}\right)^{2i+1} \phi(x) \quad . \quad . \quad . \quad . \quad . \quad (26);$$

for this theorem has been proved to be true when $\phi(x)$ is of the form $Ax^a+Bx^b+Cx^c+$&c.; and as it merely asserts an identical relation between the derived functions of $\phi(x)$, it must hold good universally, since the truth of such a relation could not be dependent on the fact of whether $\phi(x)$ was or was not expressible in any particular form.

33. The general property upon which the theorem (26) depends is that the symbols of operation

$$x^{1-a}\frac{d}{dx}x^a, \quad x^{1-\beta}\frac{d}{dx}x^\beta, \text{ &c.}$$

are convertible as regards order*; that is to say, operating with such symbols upon $\phi(x)$, the result is the same in whatever order the operations are performed. This is evident, for

$$x^{1-a}\frac{d}{dx}x^a.x^p=(p+a)x^p,$$

so that the result of the operations upon x^p, and therefore upon $\phi(x)$, is independent of the order in which they are performed.

Now the left-hand side of (25) multiplied by x^{2i+2} is

$$\left(x^{2i+1}\frac{d}{dx}x^{-2i}.x^{2i-1}\frac{d}{dx}x^{-2i+2}\ldots x^3\frac{d}{dx}x^{-2}\right)\left(x^{2i+2}\frac{d}{dx}x^{-2i-1}.x^{2i}\frac{d}{dx}x^{-2i+1}\ldots x^2\frac{d}{dx}x^{-1}\right)x\phi(x),$$

* See CAYLEY, 'Proceedings of the London Mathematical Society,' vol. viii. (1876), p. 51, and also 'Solutions of the Cambridge Senate-House Problems and Riders' for 1878, pp. 99, 100.

and, writing the operators in a different order, this expression

$$=x^{2i+2}\frac{d}{dx}x^{-2i-1}.x^{2i+1}\frac{d}{dx}x^{-2i}.x^{2i}\frac{d}{dx}x^{-2i+1}\ \ldots\ x\frac{d}{dx}x^{-1}.x\phi(x)$$

$$=x^{2i+2}\left(\frac{d}{dx}\right)^{2i+1}\phi(x).$$

This investigation of (25) is in effect the same as that given in the last article, but the form in which the process is presented is somewhat preferable.

Denoting for the moment the operator $x^{1+n}\frac{d}{dx}x^{-n}$ by $[n]$, then

$$[a][a+b]\ldots[a+(i-1)b]\phi(x)=x^{1+a}\frac{d}{dx}x^{-a}.x^{1+a+b}\frac{d}{dx}x^{-a-b}\ \ldots\ x^{1+a+(i-1)b}\frac{d}{dx}x^{-a-(i-1)b}.\phi(x$$

$$=x^{a-b}\left(x^{b+1}\frac{d}{dx}\right)^{i}\frac{\phi(x)}{x^{a+(i-1)b}}.$$

Also, writing the operators in the reverse order,

$$[a+(i-1)b]\ldots[a+b][a]\phi(x)=x^{1+a+(i-1)b}\frac{d}{dx}x^{-a-(i-1)b}\ \ldots\ x^{1+a+b}\frac{d}{dx}x^{-a-b}.x^{1+a}\frac{d}{dx}x^{-a}.\phi(x)$$

$$=x^{a+ib}\left(x^{-b+1}\frac{d}{dx}\right)^{i}\frac{\phi(x)}{x^{a}}.$$

Thus

$$x^{a-b}\left(x^{b+1}\frac{d}{dx}\right)^{i}\frac{\phi(x)}{x^{a+(i-1)b}}=x^{a+ib}\left(x^{-b+1}\frac{d}{dx}\right)^{i}\frac{\phi(x)}{x^{a}},$$

and therefore, replacing $\frac{\phi(x)}{x^{a+(i-1)b}}$ by $\phi(x)$,

$$\left(x^{b+1}\frac{d}{dx}\right)^{i}\phi(x)=x^{(i+1)b}\left(x^{-b+1}\frac{d}{dx}\right)^{i}x^{(i-1)b}\phi(x)\quad\ldots\ldots\quad(27),$$

or, writing $i+1$ for i,

$$\left(x^{b+1}\frac{d}{dx}\right)^{i+1}\phi(x)=x^{(i+2)b}\left(x^{-b+1}\frac{d}{dx}\right)^{i+1}x^{ib}\phi(x)\,.\quad\ldots\ldots\quad(28),$$

34. Putting $b=-2$, and $\phi(x)=\frac{e^{ax}}{x}$ and e^{ax} respectively in (27) and (28), these formulæ give

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i}\frac{e^{ax}}{x}=\frac{1}{x^{2i+2}}\left(x^{3}\frac{d}{dx}\right)^{i}\frac{e^{ax}}{x^{2i-1}},$$

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=\frac{1}{x^{2i+4}}\left(x^{3}\frac{d}{dx}\right)^{i+1}\frac{e^{ax}}{x^{2i}};$$

whence

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=a\left(\frac{1}{x}\frac{d}{dx}\right)^{i}\frac{e^{ax}}{x}=\frac{1}{x^{2i+4}}\left(x^3\frac{d}{dx}\right)^{i+1}\frac{e^{ax}}{x^{2i}}=\frac{a}{x^{2i+2}}\left(x^3\frac{d}{dx}\right)^{i}\frac{e^{ax}}{x^{2i-1}}.$$

The complete integral of (1) may therefore be written also in the form

$$u=\frac{1}{x^{i+1}}\left(x^3\frac{d}{dx}\right)^{i}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x^{2i-1}}\right) \quad . \quad . \quad . \quad . \quad (29)$$

or in the form

$$u=\frac{1}{x^{i+3}}\left(x^3\frac{d}{dx}\right)^{i+1}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x^{2i}}\right) \quad . \quad . \quad . \quad . \quad (30).$$

The first of these solutions, viz. (29), is that given by BOOLE in the 'Philosophical Transactions' for 1844,* and in his 'Treatise on Differential Equations,' chap. xvii., BOOLE's process is as follows: he shows that

$$u=e^{-i\theta}(\mathrm{D}-1)(\mathrm{D}-3)\ldots(\mathrm{D}-2i+1)v$$

where

$$\mathrm{D}\text{ denotes }\frac{d}{d\theta},\qquad x=e^{\theta},\qquad v=\frac{c_1e^{ax}+c_2e^{-ax}}{x}$$

and he thence deduces that

$$u=e^{-i\theta}.e^{\theta}\mathrm{D}e^{-\theta}.e^{3\theta}\mathrm{D}e^{-3\theta}\ldots e^{(2i-1)\theta}\mathrm{D}e^{-(2i-1)\theta}v$$

$$=e^{-(i+1)\theta}(e^{2\theta}\mathrm{D})^{i}e^{-(2i-1)\theta}v=\frac{1}{x^{i+1}}\left(x^3\frac{d}{dx}\right)^{i}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x^{2i-1}}\right).$$

But if the factors are written in the reverse order, we have

$$u=e^{-i\theta}(\mathrm{D}-2i+1)\ldots(\mathrm{D}-3)(\mathrm{D}-1)v$$

$$=e^{-i\theta}.e^{(2i-1)\theta}\mathrm{D}e^{-(2i-1)\theta}\ldots e^{3\theta}\mathrm{D}e^{-3\theta}.e^{\theta}\mathrm{D}e^{-\theta}.v$$

$$=e^{(i+1)\theta}(e^{-2\theta}\mathrm{D})^{i}e^{-\theta}v$$

$$=x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right),$$

which is (19). BOOLE does not seem to have anywhere alluded to the connexion between his own form (29) and the form (19), or to have remarked that the latter was obtainable by his own method.

Putting $b=-2q$ (q unrestricted) in (27) and (28), we find

* "On a General Method in Analysis," p. 252.

$$\left(z^{-2q+1}\frac{d}{dz}\right)^i \frac{e^{\frac{a}{q}z^q}}{z^q}=\frac{1}{z^{2iq+2q}}\left(z^{2q+1}\frac{d}{dz}\right)^i \frac{e^{\frac{a}{q}z^q}}{z^{2iq-2q}},$$

$$\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1} e^{\frac{a}{q}z} =\frac{1}{z^{2iq+4q}}\left(z^{2q+1}\frac{d}{dz}\right)^{i+1} \frac{e^{\frac{a}{q}z^q}}{z^{2iq}};$$

so that the solution of RICCATI's equation may be written also in the forms

$$u=\frac{1}{z^q}\left(z^{2q+1}\frac{d}{dz}\right)^i \left(\frac{c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}}{z^{1-3q}}\right),$$

$$u=\frac{1}{z^{3q}}\left(z^{2q+1}\frac{d}{dz}\right)^{i+1} \left(\frac{c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}}{z^{1-q}}\right),$$

if $q=\frac{1}{2i+1}$, and

$$u=z^{1-q}\left(z^{2q+1}\frac{d}{dz}\right)^i \{z^{3q+1}(c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q})\},$$

$$u=z^{1-3q}\left(z^{2q+1}\frac{d}{dz}\right)^{i+1} \{z^{q+1}(c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q})\},$$

if $q=-\frac{1}{2i+1}$.

35. BOOLE's form (29) of the solution of the equation (1) can be obtained also from the definite integral (18) in art. 29; for we have

$$u=x^{p+1}\int_0^\infty\frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}d\xi=x^{-p-1}\int_0^\infty\frac{\cos a\xi}{(1+b\xi^2)^{p+1}}d\xi, \text{ if } b=x^{-2},$$

$$=\frac{1}{p}x^{-p-1}\left(\int_a^\infty da\right)^2\frac{d}{db}\int_0^\infty\frac{\cos a\xi}{(1+b\xi^2)^p}d\xi$$

$$=\frac{1}{p!}x^{-p-1}\left[\left(\int_a^\infty da\right)^2\frac{d}{db}\right]^p\int_0^\infty\frac{\cos a\xi}{1+b\xi^2}d\xi, \text{ when } p \text{ is a positive integer,}$$

$$=\tfrac{1}{2}\pi.\frac{1}{p!}x^{-p-1}\left[\left(\int_a^\infty da\right)^2\frac{d}{db}\right]^p\frac{e^{-\frac{a}{\sqrt{b}}}}{\sqrt{b}}$$

$$=\tfrac{1}{2}\pi.\frac{1}{p!}x^{-p-1}\left(\frac{d}{db}\right)^p\left(\int_a^\infty da\right)^{2p}\frac{e^{-\frac{a}{\sqrt{b}}}}{\sqrt{b}}$$

$$=\tfrac{1}{2}\pi.\frac{1}{p!}x^{-p-1}\left(\frac{d}{db}\right)^p\left\{b^{\frac{1}{2}(2p-1)}e^{-\frac{a}{\sqrt{b}}}\right\}$$

$$=\tfrac{1}{2}\pi.\frac{(-1)^p}{2^p.p!}x^{-p-1}\left(x^3\frac{d}{dx}\right)^p\frac{e^{-ax}}{x^{2p-1}},$$

leading to the complete integral

$$u=x^{-p-1}\left(x^3\frac{d}{dx}\right)^p \frac{c_1e^{ax}+c_2e^{-ax}}{x^{2p-1}}.$$

It will be observed that

$$\frac{d}{db}\int_0^\infty \frac{\cos a\xi}{(1+b\xi^2)^p}d\xi=-p\int_0^\infty \frac{\xi^2\cos a\xi}{(1+b\xi^2)^{p+1}}d\xi,$$

both integrals being finite for every positive integral value of p, and that the second integral when integrated with regard to a between the limits ∞ and a

$$=-p\int_0^\infty \frac{\xi\sin(\infty\xi)}{(1+b\xi^2)^{p+1}}d\xi+p\int_0^\infty \frac{\xi\sin(a\xi)}{(1+b\xi^2)^{p+1}}d\xi.$$

The former of these two integrals is zero, as it can be shown that $\int_0^\infty \frac{\xi\sin a\xi}{(1+b\xi^2)^{p+1}}d\xi$ diminishes as a increases, and, in the limit when a is infinite, vanishes. A similar remark applies to the second integration with regard to a. The above process does not therefore involve the assumptions, $\sin\infty=0$, $\cos\infty=0$.

36. POISSON'S theorem quoted in § V., art. 20, viz. that the definite integral (5) satisfies the differential equation (6) shows that RICCATI'S equation

$$\frac{d^2u}{dz^2}-a^2z^{2q-2}u=0 \quad \ldots\ldots\ldots\ldots \quad (4)$$

is satisfied by the definite integral

$$u=\int_0^\infty e^{-x^{2q}-\frac{a^2z^{2q}}{4q^2x^{2q}}}dx \quad \ldots\ldots\ldots\ldots \quad (31).$$

Putting $\frac{a^2z^{2q}}{4q^2}=\alpha$, and transforming the integral by taking $x^{2q}=\alpha x'^2$, we find that

$$u=\frac{1}{q}\alpha^{\frac{1}{2q}}\int_0^\infty x^{\frac{1}{q}-1}\, e^{-\alpha x^2-\frac{1}{x^2}}dx,$$

which, if $\frac{1}{q}-1=2i$,

$$=\frac{1}{q}(-)^i\alpha^{\frac{1}{2q}}\left(\frac{d}{d\alpha}\right)^i\int_0^\infty e^{-\alpha x^2-\frac{1}{x^2}}dx$$

$$=\frac{1}{q}(-)^i\alpha^{\frac{1}{2q}}\frac{\sqrt{\pi}}{2}\left(\frac{d}{d\alpha}\right)^i\frac{e^{-2\sqrt{\alpha}}}{\sqrt{\alpha}}=\frac{1}{q}(-)^{i+1}\alpha^{\frac{1}{2q}}\frac{\sqrt{\pi}}{2}\left(\frac{d}{d\alpha}\right)^{i+1}e^{-2\sqrt{\alpha}};$$

so that the differential equation is satisfied by

$$u=z\left(\frac{d}{d\alpha}\right)^{i+1}e^{-2\sqrt{\alpha}} \quad \ldots\ldots\ldots\ldots \quad (32).$$

Similarly, by transforming (31) to the form,

$$u=\frac{1}{q}\int_0^\infty x^{\frac{1}{q}-1}\, e^{-x^2-\frac{a}{x^2}}dx,$$

we find that the differential equation is satisfied by

$$u=\left(\frac{d}{d\alpha}\right)^{i+1}e^{-2\sqrt{\alpha}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

if $\frac{1}{q}-1=-2i-2$ that is, if $q=-\frac{1}{2i+1}$.

The formulæ (32) and (33), on substituting for α its value in terms of z, lead at once to the solutions given in art. 29.

This method was applied by POISSON* to show that the equation (4) is integrable when $q=\pm\frac{1}{2i+1}$.

The formula (25) of art. 32 may be easily deduced from the equation

$$\int_0^\infty e^{-ax^2-\frac{b}{x^2}}dx=\frac{\sqrt{\pi}}{2\sqrt{a}}e^{-2\sqrt{(ab)}};$$

for we have

$$\int_0^\infty x^{2i}e^{-ax^2-\frac{b}{x^2}}dx=\frac{\sqrt{\pi}}{2}\left(-\frac{d}{da}\right)^i\frac{e^{-2\sqrt{(ab)}}}{\sqrt{a}},$$

and also

$$=\frac{\sqrt{\pi}}{2}\left(-\int_\infty^b db\right)^i\frac{e^{-2\sqrt{(ab)}}}{\sqrt{a}},$$

so that

$$\left(\int_\infty^b db\right)^i e^{-2\sqrt{(ab)}}=\sqrt{a}\left(\frac{d}{da}\right)^i\frac{e^{-2\sqrt{(ab)}}}{\sqrt{a}},$$

whence †

$$\sqrt{b}\left(\int_\infty^b db\right)^i e^{-2\sqrt{(ab)}}=-\sqrt{a}\left(\frac{d}{da}\right)^{i+1}e^{-2\sqrt{(ab)}} \quad . \quad . \quad . \quad . \quad . \quad (34).$$

* 'Journal de l'École Polytechnique,' vol. ix., pp. 236, 237. POISSON'S investigation is reproduced in DE MORGAN'S 'Differential and Integral Calculus,' pp. 703, 704. The formulæ (32) and (33) are obtained in the same manner as in the text from the integral (31), and the solutions in art. 30 are deduced from them, in a paper "On RICCATI'S Equation" ('Quarterly Journal of Mathematics,' vol. xi., 1871, pp. 267–273).

† In the paper "On RICCATI'S equation," referred to in the preceding note, the following two formulæ occur

$$\left\{\int^\infty d\beta\right\}^i e^{-2\sqrt{(\alpha\beta)}}=\sqrt{\alpha}\left(-\frac{d}{d\alpha}\right)^{i+1}e^{-2\sqrt{(\alpha\beta)}}, \qquad \sqrt{\alpha}\left\{\int^\infty d\alpha\right\}^{i-1}e^{-2\sqrt{(\alpha\beta)}}=(-)^{i-1}\left(\frac{d}{d\beta}\right)^i e^{-2\sqrt{(\alpha\beta)}}.$$

These are inaccurate owing to the omission of the factor $\sqrt{\beta}$ in both, and a wrong sign in the latter; when these corrections are made, both become identical with (34). The formula (34) is given, and (35) is deduced from it, in a paper "Sur une propriété de la fonction $e^{\sqrt{x}}$" ('Nouvelle Correspondance Mathématique,' vol. ii. (1876), p. 240).

Now, identically, ϕ denoting any function,

$$a^i\left(\frac{d}{da}\right)^i\phi(ab)=b^i\left(\frac{d}{db}\right)^i\phi(ab),$$

and therefore (34) may be transformed into

$$a^{i+\frac{1}{2}}\left(\int_\infty^b db\right)^i e^{-2\sqrt{(ab)}}=-b^{i+\frac{1}{2}}\left(\frac{d}{db}\right)^{i+1}e^{-2\sqrt{(ab)}},$$

which, putting $a=1$, becomes

$$\left(\int_\infty^b db\right)^i e^{-2\sqrt{b}}=-b^{i+\frac{1}{2}}\left(\frac{d}{db}\right)^{i+1}e^{-2\sqrt{b}}.$$

Therefore

$$\left(\frac{d}{db}\right)^i b^{i+\frac{1}{2}}\left(\frac{d}{db}\right)^{i+1}e^{-2\sqrt{b}}=-e^{-2\sqrt{b}};$$

replacing $e^{-2\sqrt{b}}$ by $e^{\sqrt{b}}$, this formula becomes

$$2^{2i+1}\left(\frac{d}{db}\right)^i b^{i+\frac{1}{2}}\left(\frac{d}{db}\right)^{i+1}e^{\sqrt{b}}=e^{\sqrt{b}},$$

whence, taking $b=a^2x^2$,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=a^{2i+1}e^{ax} \quad \ldots \quad (35).$$

As was shown in art. 32, this is a particular case of the more general formula

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}\phi(x)=\left(\frac{d}{dx}\right)^{2i+1}\phi(x);$$

and this formula itself admits of generalisation, as it can be proved that

$$\left(\frac{1}{x^2}\frac{d}{dx}\right)^i x^{3i+1}\left(\frac{1}{x^2}\frac{d}{dx}\right)^i x^{3i+1}\left(\frac{1}{x^2}\frac{d}{dx}\right)^{i+1}\phi(x)=\left(\frac{d}{dx}\right)^{3i+1}\phi(x),$$

$$\left(\frac{1}{x^3}\frac{d}{dx}\right)^i x^{4i+1}\left(\frac{1}{x^3}\frac{d}{dx}\right)^i x^{4i+1}\left(\frac{1}{x^3}\frac{d}{dx}\right)^i x^{4i+1}\left(\frac{1}{x^3}\frac{d}{dx}\right)^{i+1}\phi(x)=\left(\frac{d}{dx}\right)^{4i+1}\phi(x),$$

and generally, r being any positive integer,

$$\left\{\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^i x^{ri+1}\right\}^{r-1}\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^{i+1}\phi(x)=\left(\frac{d}{dx}\right)^{ri+1}\phi(x).$$

These formulæ are obtained in a paper "On Certain Identical Differential Relations," published in the 'Proceedings of the London Mathematical Society,' vol. viii. (1876), pp. 47–51.

37. As mentioned in art. 34, the integral

$$u=\frac{1}{x^{i+1}}\left(x^3\frac{d}{dx}\right)^i\left(\frac{c_1e^{ax}+c_2e^{ax}}{x^{2i-1}}\right)$$

of the differential equation (1) was first given by BOOLE in the 'Philosophical Transactions' for 1844.

The integral

$$u=x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^i\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

is due to Mr. GASKIN, and was in effect given by him in a problem set in the Cambridge Senate House Examination for 1839. The problem is as follows*:

"If m be the greatest root of the equation $m^2+m=a$,

$$\mathrm{C}d^m_{r=n^2}\left\{\frac{\cos(x\sqrt{r}+\alpha)}{x^m\sqrt{r}}\right\} \quad \text{or} \quad \mathrm{C}x^{m+1}\left(\int_{r=n}-\int_{r=-n}\right)(r^2-n^2)^m\cos(rx+\alpha)$$

are general values of y in the equation $d_x^2y+\left(n^2-\frac{a}{x^2}\right)y=0$ according as m is an integer or fraction: and in the first case $(d_x^2+n^2)^{m+1}u=0$ where $u=yx^m$; apply the first or third result to solve the equation

$$d_x^2y+\left(n^2-\frac{6}{x^2}\right)y=0."$$

Thus Mr. GASKIN's theorem is that the solution of

$$\frac{d^2u}{dx^2}+a^2u=\frac{p(p+1)}{x^2}u$$

is

$$u=\mathrm{C}x^{-p}\left(\frac{d}{dr}\right)^p\frac{\cos(x\sqrt{r}+\alpha)}{\sqrt{r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

where r is to be put equal to a^2 after the performance of the differentiations, p being a positive integer, and that in general

$$u=\mathrm{C}x^{p+1}\int_{-a}^{a}(r^2-a^2)^p\cos(rx+\alpha)dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38),$$

p being any positive quantity.

* The problem forms the second part of Question 8 of the paper set on the afternoon of Tuesday, January 8, 1839 ('Cambridge University Calendar,' 1839, p. 319).

The form (37) is readily identified with (36), for, from (37),

$$u=\mathrm{C}x^{-p}\left(\frac{1}{a}\frac{d}{da}\right)^p\frac{\cos(xa+\alpha)}{a}$$

$$=\mathrm{C}x^{-p}.x^{2p+1}\left(\frac{1}{\xi}\frac{d}{d\xi}\right)^p\frac{\cos(\xi+\alpha)}{\xi},\ \text{if}\ \xi=ax,$$

$$=\mathrm{C}x^{p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^p\frac{\cos(ax+\alpha)}{x}.$$

A method of proving the theorems contained in Mr. GASKIN'S question is given in HYMERS'S 'Treatise on Differential Equations, and on the Calculus of Finite Differences' (Cambridge, 1839) pp. 83–85. The result (38) is there verified by showing that

$$v=\int_{-a}^{a}(r^2-a^2)^p\cos(xr+\alpha)dr$$

satisfies the differential equation

$$\frac{d^2v}{dx^2}+\frac{2p+2}{x}\frac{dv}{dx}+a^2v=0\ ;$$

and it is remarked that (37) may be verified in a similar manner by showing that

$$v=\left(\frac{d}{dr}\right)^p\frac{\cos(x\sqrt{r}+\alpha)}{\sqrt{r}},$$

r being put equal to a^2, satisfies

$$\frac{d^2v}{dx^2}-\frac{2p}{x}\frac{dv}{dx}+a^2v=0.^*$$

The integral (37) was subsequently obtained by R. LESLIE ELLIS by a different process in the 'Cambridge Mathematical Journal,'† vol. ii., p. 195 (February, 1841). A full account of ELLIS'S method, with its application to the equation in question, is given in DE MORGAN'S 'Differential and Integral Calculus,' pp. 701–703.

In a paper, "Remarques sur l'équation $y''+\frac{m}{x}y'+ny=0$" ('LIOUVILLE'S Journal,' vol. xi., 1846, pp. 338–340), M. LEBESGUE proved that the integrals of the equations

$$\frac{d^2y}{dx^2}+\frac{2i}{x}\frac{dy}{dx}+ny=0,$$

* In the second edition (1858) of HYMERS'S work, only the proof that (38) satisfies the differential equation is given (p. 128), no reference being made to Mr. GASKIN'S other result. An account of BOOLE'S solution and method, taken from the 'Philosophical Transactions' for 1844, is however introduced on pp. 99–106.

† "On the Integration of Certain Differential Equations," pp. 169–177, 193–201.

and

$$\frac{d^2y}{dx^2}-\frac{2i}{x}\frac{dy}{dx}+ny=0$$

are respectively

$$y=\frac{1}{x}\left\{\left\{\frac{1}{x}\left[\frac{1}{x}\cdots\left(\frac{1}{x}\nu'\right)'\right]'\cdots\right\}'\right\},$$

and

$$y=x^{2i}\left\{\left\{\frac{1}{x}\left[\frac{1}{x}\cdots\left(\frac{1}{x}\nu'\right)'\right]'\cdots\right\}'\right\},$$

where $\nu=c\sin x\sqrt{n}+c_1\cos x\sqrt{n}$, and the former of the two expressions involves i differentiations and the latter $i+1$.

In the 'Philosophical Magazine'* for May, 1856, Mr. BENJAMIN WILLIAMSON obtained by a symbolic method the integrals of the differential equations

$$\left(\mathrm{D}^2-\frac{2i}{x}\mathrm{D}+a^2\right)y=0,\quad \left(\mathrm{D}^2+\frac{2(i+1)}{x}\mathrm{D}+a^2\right)y=0$$

in the respective forms

$$y=\mathrm{A}\left(\frac{d}{da}a^{-1}\right)^i\cos(ax+\alpha),\quad y=\mathrm{A}x^{-2i+1}\left(\frac{d}{da}a^{-1}\right)^i\cos(ax+\alpha),$$

and that of the equation

$$\left(\mathrm{D}^2-\frac{i(i+1)}{x^2}+a^2\right)y=0$$

in the form

$$y=\mathrm{A}x^{-i}\left(\frac{d}{da}a^{-1}\right)^n\cos(ax+\alpha);$$

and in the 'Philosophical Transactions'† for 1857 the late Professor DONKIN obtained, also by a symbolic method, the integral of this last equation in the form

$$y=x^i\left(\mathrm{D}\frac{1}{x}\right)^i(c_1\sin ax+c_2\cos ax).$$

* "On the Solution of Certain Differential Equations" ('Philosophical Magazine,' Fourth series, vol. xi., pp. 364–371).

† "On the Equation of LAPLACE'S Functions, &c.," vol. 147, p. 44. A proof that the integral of the partial differential equation $\frac{1}{a^2}\frac{d^2u}{dt^2}=\frac{d^2u}{dr^2}+\frac{2}{r}\frac{du}{dr}-\frac{i(i+1)}{r^2}u$, which is a simple transformation of (1), may be presented in the form $u=r^i\left(\frac{1}{r}\frac{d}{dr}\right)^i\frac{\phi(r+at)+\psi(r-at)}{r}$ is given by Professor C. NIVEN in the 'Solutions of the Senate-House Problems and Riders' for 1878, pp. 158, 159.

38. Taking the differential equation in the form (1), which has been adopted as the standard form in this memoir, it may be observed that, although the integrals

$$u = x^i \left(\frac{d}{dx}\frac{1}{x}\right)^i (c_1 e^{ax} + c_2 e^{-ax})$$

and

$$u = x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^i \left(\frac{c_1 e^{ax} + c_2 e^{-ax}}{x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

have thus been given more than once by different mathematicians, the slightly modified form

$$u = x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1} (c_1 e^{ax} + c_2 e^{-ax})$$

seems scarcely to have been noticed.* It was this form which led me to the solution in § II. as follows; if x^2 is written for ξ after the performance of the differentiations, then

$$\begin{aligned} x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1} e^{ax} &= 2^{i+1} x^{i+1}\left(\frac{d}{d\xi}\right)^{i+1} e^{a\sqrt{\xi}} \\ &= 2^{i+1} \,.\, i! \,.\, x^{i+1} \,.\, \text{coefficient of } h^{i+1} \text{ in } e^{h\frac{d}{d\xi}} . e^{a\sqrt{\xi}} \\ &= 2^{i+1} \,.\, i! \,.\, x^{i+1} \,.\, \text{coefficient of } h^{i+1} \text{ in } e^{a\sqrt{(\xi+h)}} \\ &= 2^{i+1} \,.\, i! \,.\, x^{i+1} \,.\, \text{coefficient of } h^{i+1} \text{ in } e^{a\sqrt{(x^2+h)}} \\ &= 2^{i+1} \,.\, i! \,.\, \text{coefficient of } h^{i+1} \text{ in } e^{a\sqrt{(x^2+xh)}}. \end{aligned}$$

In my paper "On a Differential Equation allied to RICCATI'S" ('Quarterly Journal,' vol. xii., 1872, p. 136), I deduced by this method from the form (39), which is the same as (19) of art. 29, that the solution of (1) was

$$u = x^{i+1} \,.\, \text{coefficient of } h^i \text{ in } \frac{c_1 e^{a\sqrt{(x^2+h)}} + c_2 e^{-a\sqrt{(x^2+h)}}}{\sqrt{(x^2+h)}},$$

but I did not then remark the far more simple form

$$u = \text{coefficient of } h^{i+1} \text{ in } c_1 e^{a\sqrt{(x^2+xh)}} + c_2 e^{-a\sqrt{(x^2+xh)}}.$$

39. It is interesting to connect Mr. GASKIN'S definite-integral solution (38) of art. 37 with that given in art. 26. The latter is

$$u = x^{p+1}\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{p+1}} d\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40),$$

* The integral is however in effect expressed in this form in EARNSHAW'S 'Partial Differential Equations' (1871), p. 92.

p being supposed to be any positive quantity; and the process of verifying that this is a solution of the equation is as follows. By actual differentiation, we have, as in art. 26,

$$\frac{d^2u}{dx^2}-\frac{p(p+1)}{x^2}u=2(p+1)x^{p+1}\int_0^\infty \{x^2-(2p+3)\xi^2\}\frac{\cos a\xi}{(x^2+\xi^2)^{p+3}}d\xi \quad . \quad . \quad (41),$$

and, by a double integration by parts,

$$\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}d\xi=\left[\frac{1}{a}\frac{\sin a\xi}{(x^2+\xi^2)^{p+1}}-\frac{2(p+1)\xi}{a^2}\frac{\cos a\xi}{(x^2+\xi^2)^{p+2}}\right]_0^\infty$$

$$+\frac{2(p+1)}{a^2}\int_0^\infty \{x^2-(2p+3)\xi^2\}\frac{\cos a\xi}{(x^2+\xi^2)^{p+3}}d\xi \quad . \quad . \quad . \quad (42).$$

The integral (40) therefore satisfies the differential equation, since the quantity in square brackets vanishes between the limits of integration.

If these limits had been any quantities α, β independent of x, instead of 0, ∞, we should have obtained a result corresponding to (41), but the quantity in square brackets in (42) would not have vanished. Replacing $p+1$ by $-p$, it is clear that

$$u=x^{-p}\int_\alpha^\beta (x^2+\xi^2)^p \cos a\xi d\xi$$

will satisfy the differential equation if α, β can be so chosen that

$$\left[\frac{1}{a}(x^2+\xi^2)^p \sin a\xi+\frac{2p\xi}{a^2}(x^2+\xi^2)^{p-1}\cos a\xi\right]_\alpha^\beta$$

is zero. This would be the case if* $\beta=xi'$, $\alpha=-xi'$, but these values of α, β are inadmissible as they are not independent of x.

Transforming now the integral in (40) by the substitution $a\xi=xt$, we have

$$x^{p+1}\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}d\xi=a^{2p+1}x^{-p}\int_0^\infty \frac{\cos xt}{(a^2+t^2)^{p+1}}dt,$$

and therefore

$$u=x^{-p}\int_0^\infty \frac{\cos xt}{(a^2+t^2)^{p+1}}dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

also satisfies the differential equation.

To verify this, we find by differentiation

$$\frac{d^2u}{dx^2}-\frac{p(p+1)}{x^2}u=2px^{-p-1}\int_0^\infty \frac{t\sin xt}{(a^2+t^2)^{p+1}}dt-x^{-p}\int_0^\infty \frac{t^2\cos xt}{(a^2+t^2)^{p+1}}dt \quad . \quad . \quad . \quad . \quad (44)$$

* As i denotes a positive integer in this memoir, i' is used to denote $\sqrt{(-1)}$.

and, by integration by parts,

$$2p\int_0^\infty \frac{t\sin xt}{(a^2+t^2)^{p+1}}dt=\left[-\frac{\sin xt}{(a^2+t^2)^p}\right]_0^\infty+\int_0^\infty \frac{x\cos xt}{(a^2+t^2)^p}dt \quad . \quad . \quad . \quad . \quad . \quad (45),$$

whence the right-hand member of (44)

$$=x^{-p}\int_0^\infty\left\{\frac{\cos xt}{(a^2+t^2)^p}-\frac{t^2\cos xt}{(a^2+t^2)^{p+1}}\right\}dt$$

$$=x^{-p}a^2\int_0^\infty\frac{\cos xt}{(a^2+t^2)^{p+1}}dt=a^2u.$$

If the limits were α, β the differential equation would still be satisfied if the quantity in square brackets in (45) vanished between these limits. This is not the case for any other values of α and β besides 0 and ∞, but if in (43) $p+1$ is replaced by $-p$, so that the integral is

$$u=x^{p+1}\int_\beta^\alpha(a^2+t^2)^p\cos xtdt,$$

then the quantity in square brackets $=-(a^2+t^2)^{p+1}\sin xt$, which vanishes when $t=\pm ai'$, and therefore the differential equation is satisfied by the integral

$$u=x^{p+1}\int_{-ai'}^{ai'}(a^2+t^2)^p\cos xtdt.$$

Since in this case the quantity in square brackets vanishes in virtue of the factor $(a^2+t^2)^{p+1}$, we may replace $\cos xt$ by $\cos(xt+\alpha)$, α being any constant, so that the solution of the differential equation may be written

$$u=Cx^{p+1}\int_{-ai'}^{ai'}(a^2+t^2)^p\cos(xt+\alpha)dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (46).$$

If in the differential equation a^2 be replaced by $-a^2$, this integral becomes

$$u=Cx^{p+1}\int_{-a}^{a}(t^2-a^2)^p\cos(xt+\alpha)dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47),$$

which is Mr. GASKIN's formula (38).

40. This is not however, as stated by Mr. GASKIN, the general integral of the differential equation, as it in fact contains only one arbitrary constant. For, evidently,

$$\int_{-a}^{a}(t^2-a^2)^p\sin x^tdt=0,$$

so that the introduction of the constant α does not increase the generality of the solution.

Returning to the integral (46), we find, on putting $\alpha=0$ and transforming the integral by the substitution $t=i'v$,

$$u=\mathrm{C}x^{p+1}\int_{-a}^{a}(v^2-a^2)^p(e^{xv}+e^{-xv})dv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

$$=\mathrm{C}x^{p+1}\left(\frac{d^2}{dx^2}-a^2\right)^p\int_{-a}^{a}(e^{xv}+e^{-xv})dv$$

$$=\mathrm{C}x^{p+1}\left(\frac{d^2}{dx^2}-a^2\right)^p\left(\frac{e^{ax}-e^{-ax}}{x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

Now, as will be shown in the next article,

$$\left(\frac{d^2}{dx^2}-a^2\right)^i\frac{e^{ax}}{x}=(-)^i2^i.i!\left(\frac{1}{x}\frac{d}{dx}\right)^i\frac{e^{ax}}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

so that the particular integral (49) is equivalent to

$$u=\mathrm{C}x^{p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{p+1}(e^{ax}-e^{-ax}).$$

The complete solution of the differential equation is, by art. 29,

$$u=x^{p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{p+1}(c_1e^{ax}+c_2e^{-ax}),$$

which may therefore be written

$$u=x^{p+1}\left(\frac{d^2}{dx^2}-a^2\right)^p\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right)$$

$$=x^{p+1}\left(\frac{d^2}{dx^2}-a^2\right)^p\left(c_1\int_{-\infty}^{a}e^{xv}dv+c_2\int_{\infty}^{a}e^{-xv}dv\right)$$

$$=x^{p+1}\left(c_1\int_{-\infty}^{a}(v^2-a^2)^pe^{xv}dv+c_2\int_{\infty}^{a}(v^2-a^2)^pe^{-xv}dv\right).$$

This is the complete solution in the form corresponding to (48).

41. To prove the relation (50), let

$$v=\int_0^\infty e^{-a^2x^2-\frac{b^2}{x^2}}dx=\frac{\sqrt{\pi}}{2a}e^{-2ab},$$

then, denoting for the sake of brevity $a^2x^2+\frac{b^2}{x^2}$ by w,

$$\frac{dv}{da}=\int_0^\infty -2ax^2e^{-w}dx,$$

whence

$$\left(-\frac{1}{2a}\frac{d}{da}\right)v=\int_0^\infty x^2e^{-w}dx$$

and therefore

$$\left(-\frac{1}{2a}\frac{d}{da}\right)^i v=\int_0^\infty x^{2i}e^{-w}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51).$$

Again,

$$\frac{d^2v}{da^2}=\int_0^\infty(-2x^2+4a^2x^4)e^{-w}dx,$$

and

$$\int_0^\infty 4a^2x^4e^{-w}dx=\left[-e^{-a^2x^2}2x^3e^{-\frac{b^2}{x^2}}\right]_0^\infty+\int_0^\infty e^{-a^2x^2}\frac{d}{dx}(2x^3e^{-\frac{b^2}{x^2}})dx$$

$$=\int_0^\infty (6x^2+4b^2)e^{-w}dx\,;$$

therefore

$$\frac{d^2v}{da^2}=\int_0^\infty (4x^2+4b^2)e^{-w}dx,$$

whence

$$\left(\frac{d^2}{da^2}-4b^2\right)v=4\int_0^\infty x^2e^{-w}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52).$$

If instead of the integral v we start with the integral

$$v_i=\int_0^\infty x^{2i}e^{-w}dx,$$

we have

$$\frac{d^2v_i}{da^2}=\int_0^\infty(-2x^{2i+2}+4a^2x^{2i+4})e^{-w}dx,$$

and, integrating the second term by parts as before, we find

$$\frac{d^2v_i}{da^2}=\int_0^\infty \{(4i+4)x^{2i+2}+4b^2x^{2i}\}e^{-w}dx,$$

so that

$$\left(\frac{d^2}{da^2}-4b^2\right)v_i=4(i+1)\int_0^\infty x^{2i+2}e^{-w}dx=4(i+1)v_{i+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

Thus from (52) and (53)

$$\left(\frac{d^2}{da^2}-4b^2\right)^i v=4^i.i!\int_0^\infty x^{2i}e^{-w}dx$$

$$=4^i.i!\left(-\frac{1}{2a}\frac{d}{da}\right)^i v, \text{ from (51)},$$

which is the relation (50).

It follows from (50) in connexion with (21) of art. 31 that

$$\left(\frac{d^2}{dx^2}-a^2\right)^i\frac{e^{ax}}{x}=(-)^i\frac{(2a)^i}{x^{i+1}}i!\left\{1-\frac{(i+1)i}{2}\frac{1}{ax}+\frac{(i+2)(i+1)i(i-1)}{2.4}\frac{1}{a^2x^2}-\&\text{c.}\right\}e^{ax}. \quad (54),$$

a formula given by HARGREAVE in the 'Philosophical Transactions'* for 1848, p. 34.

42. In the paper just referred to HARGREAVE obtained by a symbolic process the solution of the differential equation in the form

$$u=x^{i+1}(\text{D}^2-a^2)\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right),$$

and thence, by (54), deduced the solution in the expanded form.

HARGREAVE also gives on p. 45 of his memoir the complete solution of the equation

$$\frac{d^2u}{dx^2}+\frac{2m}{x}-a^2x=0$$

in the form

$$u=c_1\int_{-a}^{a}(z^2-a^2)^{m-1}e^{xz}dz+c_2x^{-2m+1}\int_{-a}^{a}(z^2-a^2)^{-m}e^{xz}dz$$

$$=c_1\int_{-1}^{1}(z^2-1)^{m-1}e^{axz}dz+c_2x^{-2m+1}\int_{-1}^{1}(z^2-1)^{-m}e^{axz}dz. \quad . \quad . \quad . \quad . \quad (55).$$

One or other of the definite integrals in (55) is however always infinite, except when m lies between 0 and 1.

In the case of the differential equation (1), this solution becomes

$$u=c_1x^{-p}\int_{-1}^{1}(z^2-1)^{-p-1}e^{axz}+c_2x^{p+1}\int_{-1}^{1}(z^2-1)^pe^{axz}dz,$$

or, as it may be written more conveniently,

$$u=c_1x^{-p}\int_{-1}^{1}(1-z^2)^{-p-1}e^{axz}+c_2x^{p+1}\int_{-1}^{1}(1-z^2)^pe^{axz}dz.$$

* "On the Solution of Linear Differential Equations," pp. 31–54.

It is easy to connect these definite integrals with the series U and V of art. 3, for

$$\int_{-1}^{1}(1-z^2)^{-p-1}e^{axz}dz=\int_{-1}^{1}(1-z^2)^{-p-1}\left(1+axz+\frac{a^2x^2z^2}{2!}+\frac{a^3x^3z^3}{3!}+\&\text{c.}\right)dz$$

$$=2\int_{0}^{1}(1-z^2)^{-p-1}\left(1+\frac{a^2x^2z^2}{2!}+\frac{a\,x^4z^4}{4!}+\&\text{c.}\right)dz$$

$$=\frac{\Gamma(-p)\Gamma(\frac{1}{2})}{\Gamma(-p+\frac{1}{2})}+\frac{a^2x^2}{2!}\,\frac{\Gamma(-p)\Gamma(\frac{3}{2})}{\Gamma(-p+\frac{3}{2})}+\frac{a^4x^4}{4!}\,\frac{\Gamma(-p)\Gamma(\frac{5}{2})}{\Gamma(-p+\frac{5}{2})}+\&\text{c.}$$

$$=\frac{\Gamma(-p)\Gamma(\frac{1}{2})}{\Gamma(-p+\frac{1}{2})}\left\{1+\frac{a^2x^2}{2!}\,\frac{\frac{1}{2}}{-p+\frac{1}{2}}+\frac{a^4x^4}{4!}\,\frac{\frac{1}{2}\cdot\frac{3}{2}}{(-p+\frac{1}{2})(-p+\frac{3}{2})}+\&\text{c.}\right\}$$

$$=\sqrt{\pi}.\frac{\Gamma(-p)}{\Gamma(-p+\frac{1}{2})}\left\{1-\frac{1}{p-\frac{1}{2}}\,\frac{a^2x^2}{2^2}+\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\,\frac{a^4x^4}{2^4.2!}-\&\text{c.}\right\},$$

whence

$$x^{-p}\int_{-1}^{1}(1-z^2)^{-p-1}e^{axz}dz=\sqrt{\pi}.\frac{\Gamma(-p)}{\Gamma(-p+\frac{1}{2})}\text{U}.$$

Similarly

$$x^{p+1}\int_{-1}^{1}(1-z^2)^{p}e^{axz}dz\quad=\sqrt{\pi}.\frac{\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\;\text{V};$$

and we thus obtain expressions for U and V as definite integrals, taken between the limits 1 and -1, for all values of p for which the integrals are finite.

§ VII.

Connexion with BESSEL'S *Functions.* Arts. 43–48.

43. If the differential equation (1) is transformed by putting $u=x^{\frac{1}{2}}w$, it assumes the form

$$\frac{d^2w}{dx^2}+\frac{1}{x}\,\frac{dw}{dx}-a^2w-(p+\tfrac{1}{2})^2\,\frac{w}{x^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56).$$

The equation of BESSEL'S Functions is

$$\frac{d^2w}{dx^2}+\frac{1}{x}\,\frac{dw}{dx}+\left(1-\frac{\nu^2}{x^2}\right)w=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),$$

so that (56) becomes identical with (57) if

$$a=\sqrt{(-1)}=i', \qquad p+\tfrac{1}{2}=\nu.$$

We may therefore pass from the solutions of the equation (1) to the solutions of BESSEL's equation (57) by multiplying by $x^{-\frac{1}{2}}$ and putting $a=\sqrt{(-1)}$, $p=\nu-\frac{1}{2}$.

44. The BESSEL's Function, $J^{\nu}(x)$, may be defined for real values of ν greater than $-\frac{1}{2}$ by either of the formulæ

$$J^{\nu}(x)=\frac{x^{\nu}}{2^{\nu}\Gamma(\nu+1)}\left\{1-\frac{x^2}{2(2\nu+2)}+\frac{x^4}{2.4(2\nu+2)(2\nu+4)}-\&c.\right\} \quad . \quad . \quad . \quad (58),$$

$$J^{\nu}(x)=\frac{1}{\sqrt{\pi}}\frac{x^{\nu}}{2^{\nu}\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}e^{i'xu}(1-u^2)^{\nu-\frac{1}{2}}du \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

where i' denotes, as throughout, $\sqrt{(-1)}$.

Comparing (58) with the expression V in art. 3, we see that if $\nu=p+\frac{1}{2}$, and if a is replaced by i', the series in the two formulæ become identical, the exact relation between V and BESSEL's Function being

$$V=Ax^{\frac{1}{2}}J^{p+\frac{1}{2}}(i'ax),$$

where A denotes the constant

$$\left(\frac{2}{i'a}\right)^{p+\frac{1}{2}}\Gamma(p+\tfrac{3}{2})$$

and p is supposed to be positive.

The formula (59) corresponds to Mr. GASKIN's definite integral solution (38) or to one of the definite integrals in HARGREAVE's solution (55).

45. It is known that $J^{\nu}(x)$ may be exhibited as the sum of two series multiplied respectively by $\sin x$ and $\cos x$, viz.*

$$J^{\nu}(x)=\frac{x^{\nu}}{2^{\nu}\Gamma(\nu+1)}(A\cos x+B\sin x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60),$$

where

$$A=1-\frac{2\nu+3}{2\nu+2}\frac{x^2}{2!}+\frac{(2\nu+5)(2\nu+7)}{(2\nu+2)(2\nu+4)}\frac{x^4}{4!}-\frac{(2\nu+7)(2\nu+9)(2\nu+11)}{(2\nu+2)(2\nu+4)(2\nu+6)}\frac{x^6}{6!}+\&c.,$$

$$B=x-\frac{2\nu+5}{2\nu+2}\frac{x^3}{3!}+\frac{(2\nu+7)(2\nu+9)}{(2\nu+2)(2\nu+4)}\frac{x^5}{5!}-\frac{(2\nu+9)(2\nu+11)(2\nu+13)}{(2\nu+2)(2\nu+4)(2\nu+6)}\frac{x^7}{7!}+\&c.$$

* LOMMEL's 'Studien über die Bessel'schen Functionen' (1868), p. 17, or TODHUNTER's 'Treatise on LAPLACE's Functions, LAMÉ's Functions, and BESSEL's Functions' (1875), p. 292.

This formula may be written

$$J^{\nu}(x)=\frac{x^{\nu}}{2^{\nu+1}\Gamma(\nu+1)}\left\{\left(1-i'x+\frac{2\nu+3}{2\nu+2}\frac{i'^2x^2}{2!}-\frac{2\nu+5}{2\nu+2}\frac{i'^3x^3}{3!}+\&c.\right)e^{i'x}\right.$$

$$\left.+\left(1+i'x+\frac{2\nu+3}{2\nu+2}\frac{i'^2x^2}{2!}+\frac{2\nu+5}{2\nu+2}\frac{i'^3x^3}{3!}+\&c.\right)e^{-i'x}\right\}.$$

and the expression on the right-hand side therefore corresponds to $\frac{1}{2}(Q+S)$ where Q and S are as defined in art. 3, so that the algebraic theorem to which the two forms of BESSEL's Functions (58) and (60) lead is $V=\frac{1}{2}(Q+S)$.

46. The formula involving descending series for BESSEL's Function, $J^{\nu}(x)$ is

$$J^{\nu}(x)=\sqrt{\left(\frac{2}{\pi x}\right)}.\left\{1-\frac{(4\nu^2-1^2)(4\nu-3^2)}{1.2.(8x)^2}+\&c.\right\}\cos\left(x-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi\right)$$

$$-\sqrt{\left(\frac{2}{\pi x}\right)}.\left\{\frac{4\nu^2-1}{1.8x}-\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{1.2.3(8x)^3}+\&c.\right\}\sin\left(x-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi\right);$$

the descending series ultimately diverge for all values of ν for which they do not terminate, but the converging terms may be used for the calculation of $J^{\nu}(x)$; and this formula was in fact employed by HANSEN in the calculation of his tables of $J^0(x)$ and $J^1(x)$*. If $\nu=p+\frac{1}{2}$, p being an integer, the series terminate and we obtain a finite expression for $J^{p+\frac{1}{2}}(x)$.

Replacing the sine and cosine by their exponential values, this formula may be written

$$J^{\nu}(x)=\frac{i'^{\nu+\frac{1}{2}}}{\sqrt{(2\pi x)}}\{(-)^{\nu+\frac{1}{2}}e^{i'x}\alpha+c^{-i'x}\beta\},$$

where

$$\alpha=1-\frac{4\nu^2-1^2}{1}\frac{1}{8i'x}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{1.2}\frac{1}{(8i'x)^2}-\&c.$$

and β differs from α only in having all the terms positive.

* LOMMEL's 'Studien über die Bessel'schen Functionen,' p. 58.

Putting $\nu=p+\frac{1}{2}$, we have therefore

$$J^{p+\frac{1}{2}}(x)=\frac{1}{\sqrt{(2\pi x)}}i'^{p+1}\{(-)^{p+1}e^{i'z}\alpha_1+e^{-i'z}\beta_1\},$$

where

$$\alpha_1=1-\frac{p(p+1)}{2}\frac{1}{i'z}+\frac{(p-1)p(p+1)(p+2)}{2.4}\frac{1}{(i'x)^2}-\&c.$$

and β_1 denotes a similar series, having all the terms positive.

If p is a positive integer, this expression corresponds to $\frac{1}{2g}(R'-P')$, when the terms in R′ and P′ are written in the reverse order, as in § IV. If p is not an integer, the series, as already mentioned, are divergent, so that, strictly speaking, the formula only has a meaning when it contains a finite number of terms. An expression can however be found for the remainder after a finite number of terms, *i.e.*, for the difference between $J^{p+\frac{1}{2}}(x)$ and the sum of these terms, by means of which the use of the formula in calculation may be justified.

47. It is a known theorem in BESSEL'S Functions that if p is a positive integer, $J^p(x)$ is equal to the coefficient of z^p in the expansion of $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$; and it follows therefore, by means of the relation between V and $J^{p+\frac{1}{2}}(i'ax)$ in art. 44, that, if $p+\frac{1}{2}=$ an even positive integer $=2m$,

$$V=(-)^m\frac{2^{2m}\Gamma(2m+1)}{a^{2m}}x^{\frac{1}{2}}\times \text{ coefficient of } z^{2m} \text{ in } \cos\frac{ax}{2}\left(z-\frac{1}{z}\right),$$

and if $p+\frac{1}{2}=$ an uneven positive integer $=2m+1$,

$$V=(-)^m\frac{2^{2m+1}\Gamma(2m+2)}{a^{2m+1}}x^{\frac{1}{2}}\times \text{ coefficient of } z^{2m+1} \text{ in } \sin\frac{ax}{2}\left(z-\frac{1}{z}\right).$$

48. It was shown in § II. that the differential equation (1) was satisfied by the coefficient of h^{p+1} in the expansion of $e^{a\sqrt{(x^2+xh)}}$, and we thus find that if $\nu=p+\frac{1}{2}$, p being a positive integer, the general integral of BESSEL'S equation (57) is

$$w=x^{-\frac{1}{2}}\{c_1\times \text{ coefficient of } h^{p+1} \text{ in the expansion of } \cos\sqrt{(x^2+xh)}$$

$$+c_2 \quad\text{,,}\quad\text{,,}\quad\text{,,}\quad \sin\sqrt{(x^2+xh)}\}.$$

§ VIII.

Writings specially connected with the contents of the memoir.

[When only a portion of a paper relates to the subject of the memoir, the page-numbers refer only to this portion.]

Writings referred to in §§ I., II., III.

(i.) 1868. Cayley. "On Riccati's Equation." 'Philosophical Magazine,' Fourth series, vol. xxxvi., pp. 348–351.

The equation is written in the form

$$\frac{d^2u}{dx^2}=x^{2q-2}u,$$

and the expressions P_2, Q_2, R_2, S_2 of art. 17 are obtained by assuming series of the forms in question and equating coefficients. Two of the series terminate when q is the reciprocal of an uneven integer.

(ii.) 1869. —— "Note on the Integration of Certain Differential Equations by Series." 'Messenger of Mathematics,' First series, vol. v., pp. 77–82.

It is shown that if we have a solution

$$A\left(x^a+\frac{a_1}{b_1}x^{a+1}+\frac{a_1a_2}{b_1b_2}x^{a+2}+\&c.\right)$$

of a differential equation, and that if one of the factors in a numerator, say a_r, vanishes, then we may stop at the preceding term, the finite series so obtained being a particular integral; but that if we continue the series, notwithstanding the evanescent factor, and if at length a factor in a denominator, say $b_s(s>r)$, vanishes, then the series recommences with the term involving x^{a+s}, and we have another particular integral

$$A'\frac{0}{0}\left(x^{a+s}+\frac{a_{s+1}}{b_{s+1}}x^{a+s+1}+\frac{a_{s+1}a_{s+2}}{b_{s+1}b_{s+2}}x^{a+s+2}+\&c.\right),$$

in which $A'\frac{0}{0}$ may be replaced by a new arbitrary constant B.

(iii.) 1872. Glaisher. "On the Relations between the Particular Integrals in Cayley's Solution of Riccati's Equation." 'Philosophical Magazine,' Fourth series, vol. xliii., pp. 433–438.

The relations between U_2, V_2, P_2, Q_2, R_2, S_2 given in art. 17 are obtained. These afford an example of the principle explained in (ii.). See the introduction, p. 763.

(iv.) 1874. BACH. "De l'Integration par les Séries de l'Équation $\frac{d^2y}{dx^2}-\frac{n-1}{x}\frac{dy}{dx}=y$." 'Annales Scientifiques de l'École Normale Supérieure.' Deuxième série, vol. iii., pp. 47–68.

Detailed account, with developments, of (i.) and (iii.). In (iii.) n is written in place of $\frac{1}{q}$ and β in place of $\frac{x^q}{q}$, so that the series are reduced to the forms given in art. 16. If the differential equation is similarly transformed it becomes

$$\frac{d^2u}{d\beta^2}-\frac{n-1}{\beta}\frac{du}{d\beta}-\beta=0.$$

This is the form of the equation adopted by M. BACH, who finally deduces the series in the case of RICCATI'S equation. The form is a very convenient one. See art. 16.

(v.) 1878. GLAISHER. "Example Illustrative of a Point in the Solution of Differential Equations in Series." 'Messenger of Mathematics,' vol. viii., pp. 20–23.

In the well-known expansions quoted in art. 11, viz.

$$\{1-\surd(1-4x)\}^p=2^px^p\left\{1+px+\frac{p(p+3)}{2!}x^2+\frac{p(p+4)(p+5)}{3!}x^3+\&c.\right\}$$

$$\{1+\surd(1-4x)\}^p=2^p\left\{1-px+\frac{p(p-3)}{2!}x^2-\frac{p(p-4)(p-5)}{3!}x^3+\&c.\right\}$$

the series are such that if p is an integer, one terminates, and after a certain number of zero terms, recommences and reproduces the other. It follows therefore that the differential equation whose general integral is

$$u=c_1\{1-\surd(1-4x)\}^p+c_2\{1+\surd(1-4x)\}^p$$

must afford an example of the principle pointed out in (i.). The differential equation is found to be

$$x(1-4x)\frac{d^2u}{dx^2}+\{(4p-6)x-p+1\}\frac{du}{dx}-p(p-1)u=0,$$

and its integration in series affords the illustration referred to in the title. The note was suggested by art. 11. See art. 15.

(vi.) 1878. —— "Generalised Form of Certain Series." 'Proceedings of the London Mathematical Society,' vol. ix., pp. 197–202.

Theorems deduced from

$$\left(1-x+\frac{n+2}{n+1}\frac{x^2}{2!}+\frac{(n+2)(n+4)}{(n+1)(n+2)}\frac{x^3}{3!}-\&c.\right)e^x$$

$$=\left(1+x+\frac{n+2}{n+1}\frac{x^2}{2!}+\frac{(n+2)(n+4)}{(n+1)(n+2)}\frac{x^3}{3!}+\&c.\right)e^{-x}.$$

See art. 7.

(vii.) 1878. —— "On the Solution of a Differential Equation allied to RICCATI'S." 'British Association Report' for 1878 (Dublin), pp. 469, 470.

Proof that the coefficient of h^{i+1} in the expansion of $e^{a\sqrt{(x^2+xh)}}$ satisfies the differential equation

$$\frac{d^2u}{da^2}-a^2u=\frac{i(i+1)}{x^2}u.$$

See arts. 8, 9.

WRITINGS REFERRED TO IN § V.

(viii.) 1813. POISSON. "Mémoire sur les Intégrales Définies." 'Journal de l'École Polytechnique,' vol. ix. (cah. xvi.), pp. 236–239, 241.

It is proved that if

$$y=\int_0^\infty e^{-x^n-\frac{ba^n}{x^n}}dx,$$

then y satisfies the differential equation $\frac{d^2y}{da^2}=n^2ba^{n-2}y$, and it is deduced from this result that the equation is integrable in a finite form when $n=\frac{2}{1\pm 2i}$. See arts. 20, 36.

A relation between two definite integrals is also proved. See art. 26.

(ix.) 1872. GLAISHER. "On the Evaluation in Series of Certain Definite Integrals." 'British Association Report' for 1872 (Brighton), Transactions of the Sections, pp. 15–17.

Investigation of the formula (8) of art. 21 by the process given in arts. 21, 22.

WRITINGS REFERRED TO IN § VI.

(x.) 1839. GASKIN. Senate House Problem.

The solution of the equation

$$\frac{d^2u}{dx^2}+a^2u=\frac{p(p+1)}{x^2}u$$

is given in the forms

$$u=\mathrm{C}x^{-p}\left(\frac{d}{dr}\right)^p\frac{\cos(x\sqrt{r}+\alpha)}{\sqrt{r}},$$

r being put equal to a^2 after the differentiations, and

$$u = \mathrm{C}x^{p+1}\int_{-a}^{a}(r^2 - a^2)^p \cos(rx + \alpha)dr.$$

See arts. 37, 39.

(xi.) 1839. HYMERS. 'A Treatise on Differential Equations and on the Calculus of Finite Differences' (1839), pp. 83–85; also, second edition (1858), p. 125.

Solution of Mr. GASKIN's problem in (x.). See art. 37.

(xii.) 1841. ELLIS. "On the Integration of Certain Differential Equations," 'Cambridge Mathematical Journal,' vol. ii., pp. 193–195.

Independent investigation of the first of Mr. GASKIN's forms in (x.). See art. 37.

(xiii.) 1841. DE MORGAN. 'The Differential and Integral Calculus,' pp. 702–704. Account of ELLIS's method (see xii.) and of POISSON's determination of the integrable cases of RICCATI's equation (see viii.). See arts. 36, 37.

(xiv.) 1844. BOOLE. "On a General Method in Analysis," 'Philosophical Transactions' for 1844, pp. 251, 252.

This paper contains BOOLE's general symbolic method. The solution of the equation (1) is given in the form

$$u = \frac{1}{x^{i+1}}\left(x^3\frac{d}{dx}\right)^i \frac{c_1e^{ax} + c_2e^{-ax}}{x^{2i-1}}.$$

The general method and this solution are reproduced with only slight changes in BOOLE's 'Differential Equations,' chapter xvii. See art. 34.

(xv.) 1846. LEBESGUE. "Remarques sur l'Équation $y'' + \frac{m}{x}y' + ny = 0$," 'LIOUVILLE's Journal,' vol. xi., pp. 338, 339.

Solution of this equation in a form involving repeated differentiations with regard to x. See art. 37.

(xvi.) 1848. HARGREAVE. "On the Solution of Linear Differential Equations," 'Philosophical Transactions' for 1848, pp. 34, 35, 45.

The paper contains the general integral of (1) in the forms,

$$u = x^{i+1}(\mathrm{D}^2 - a^2)^i \frac{c_1e^{ax} + c_2e^{-ax}}{x},$$

$$u = c_1x^{-p}\int_{-1}^{1}(z^2 - 1)^{-p-1}e^{axz}dz + c_2x^{p+1}\int_{-1}^{1}(z^2 - 1)^p e^{axz}dz,$$

and a development of $(\mathrm{D}^2 - a^2)^i \frac{e^{ax}}{x}$ in a series. There are also solutions of other allied equations. See arts. 41, 42.

(xvii.) 1856. WILLIAMSON. "On the Solution of Certain Differential Equations." 'Philosophical Magazine,' Fourth series, vol. xi., pp. 364–369.

The general integral of the equation

$$\frac{d^2u}{dx^2}+a^2u=\frac{i(i+1)}{x^2}u$$

is given in the form

$$u=\mathrm{A}x^{-i}\left(\frac{d}{da}a^{-1}\right)^i \cos (ax+\alpha),$$

and the solutions of RICCATI's and several other equations are also obtained. The symbolic expressions are developed by means of the theorem

$$(\mathrm{D}a^{-1})^n=a^{-n}\mathrm{D}^n-\frac{n(n+1)}{2}a^{-(n+1)}\mathrm{D}^{n-1}+\frac{(n-1)n(n+1)(n+2)}{2.4}a^{-(n+2)}\mathrm{D}^{n-2}\ldots$$

$$\pm 1.3\ldots(2n-1)a^{-(2n-1)}(\mathrm{D}-a^{-1}),$$

of which a proof is given. See art. 37.

(xviii.) 1857. DONKIN. "On the Equation of LAPLACE's Functions, &c." 'Philosophical Transactions' for 1857, p. 44.

The integral of the equation in (xvii.) is given in the form

$$x^i\left(\mathrm{D}\frac{1}{x}\right)^i(c_1 \sin ax+c_2 \cos ax).$$

This solution occurs in a note, as an example of the application of the general method of the paper to a particular equation. See art. 37.

(xix.) 1871. GLAISHER. "On RICCATI's Equation." 'Quarterly Journal of Mathematics,' vol. xi. pp. 267–273.

By means of the definite integral (31) of art. 36, the solution of RICCATI's equation is obtained in the forms

$$u=z\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1}\left(c_1e^{\frac{1}{q}z^q}+c_2e^{-\frac{1}{q}z^q}\right), \quad \&c.$$

and the formulæ (22) and (23) of art. 31 are proved. See arts. 31, 36.

(xx.) 1872. —— "On a Differential Equation allied to RICCATI's." 'Quarterly Journal of Mathematics,' vol. xii., pp. 129–137.

The equation is (1), and the definite integral $\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{i+1}} d\xi$ is applied as in art. 29 to obtain the general integral in the form

$$u = x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^i \frac{c_1 e^{ax} + c_2 e^{-ax}}{x},$$

and also in BOOLE's form (29): the results are transformed so as to give the symbolic solution of RICCATI's equation, which is integrated also by BOOLE's method. See arts. 29, 30, 33, 34, 35, 38.

(xxi.) 1876. —— "Sur une Propriété de la Fonction $e^{\sqrt{x}}$." 'Nouvelle Correspondance Mathématique,' vol. ii., pp. 240–243, 349–350.

Proof of the theorem

$$2^{2n+1}\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{2}}\left(\frac{d}{dx}\right)^{n+1} e^{\sqrt{x}} = e^{\sqrt{x}}$$

by means of the integral

$$\int_0^\infty e^{-ax^2-\frac{b}{x^2}} dx = \frac{\sqrt{\pi}}{2\sqrt{a}} e^{-2\sqrt{(ab)}}.$$

See art. 36.

(xxii.) 1876. —— "On Certain Identical Differential Equations." 'Proceedings of the London Mathematical Society,' vol. viii., pp. 47–51.

Generalisations of the theorem in (xxi.), as for example

$$\left\{\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{r}}\right\}^{r-1}\left(\frac{d}{dx}\right)^{n+1}\phi(x^{\frac{1}{r}}) = \frac{1}{r^{rn+1}}\phi^{(rn+1)}(x^{\frac{1}{r}}),$$

and other similar results. See arts. 33, 36.

(xxiii.) 1879. —— "On a Symbolic Theorem involving Repeated Differentiations." 'Proceedings of the Cambridge Philosophical Society,' vol. iii., pp. 269–271.

The theorem is (50) of art. 40, viz.

$$\left(\frac{d^2}{dx^2} - a^2\right)^n \frac{e^{ax}}{x} = (-)^n 2^n . n!\left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{e^{ax}}{x},$$

and the proof is the same as in art. 41.